More on the infinite sum of reciprocal usual Fibonacci, Pell and higher order recurrences

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Abstract

Recently some authors [Z. Wenpeng and W. Tingting, Appl. Math. Comput. 218 (10) (2012), 6164–6167; T. Komatsu and V. Laohakosol, J. Integer Seq. 13 (5) (2010), Article 10.5.8.] computed partial infinite sums including reciprocal usual Fibonacci, Pell and generalized order-k Fibonacci numbers. In this paper we will present generalizations of earlier results by considering more generalized higher order recursive sequences with additional one coefficient parameter.

Key words: Reciprocal sums, Fibonacci-Pell numbers, higher order recurrence

1 Introduction

Let p and q be real numbers such that $p^2 + 4q \neq 0$. Define the generalized Fibonacci sequence $\{U_n(p,q)\}$, briefly $\{U_n\}$, and Lucas sequence $\{V_n(p,q)\}$, briefly $\{V_n\}$, as shown: for $n > 1$

$$\begin{aligned} U_n(p,q) &= pU_{n-1}(p,q) + qU_{n-2}(p,q), \\ V_n(p,q) &= pV_{n-1}(p,q) + qV_{n-2}(p,q), \end{aligned}$$

where $U_0 = 0,\ U_1 = 1$, and, $V_0 = 2,\ V_1 = p$, respectively. The Binet formulae for $\{U_n\}$ and $\{V_n\}$ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = \left(p \pm \sqrt{p^2 + 4q}\right)/2$. Here note that $U_n(1,1) = F_n$ (nth Fibonacci Number), $V_n(1,1) = L_n$ (nth Lucas Number) and $U_n(2,1) = P_n$ (nth Pell Number).

Ohtsuka and Nakamura [5] introduced and computed the following partial infinite sums including reciprocal Fibonacci numbers :

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \text{ is even and } n \geq 2, \\ F_{n-2} - 1 & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \tag{1}$$

and

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1 & \text{if } n \text{ is even and } n \geq 2, \\ F_{n-1}F_n & \text{if } n \text{ is odd and } n \geq 1, \end{cases}$$

where $\lfloor \cdot \rfloor$ is the floor function.

Zhang and Wang [6] gave analogue of the result (1) for the Pell numbers :

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k} \right)^{-1} \right\rfloor = \begin{cases} P_{n-1} + P_{n-2} & \text{if } n \text{ is even and } n \geq 2; \\ P_{n-1} + P_{n-2} - 1 & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

Also the same authors [7] gave the similar results for partial infinite sums including reciprocal squared-Pell numbers.

Holliday and Komatsu [1] obtained similar results for the terms of generalized Fibonacci sequence $\{U_n(p,1)\}$:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{U_k} \right)^{-1} \right\rfloor = \begin{cases} U_n - U_{n-1} & \text{if } n \text{ is even and } n \geq 2, \\ U_n - U_{n-1} - 1 & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \tag{2}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{U_k^2} \right)^{-1} \right\rfloor = \begin{cases} pU_nU_{n-1} - 1 & \text{if } n \text{ is even and } n \geq 2, \\ pU_nU_{n-1} & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

In this paper we will consider on the following type higher order recurrence sequences and then give general results similar to the above partial sums. For any positive reals p and q, we define a kth order linear recursive sequence $\{u_n(p,q,k)\}$, briefly $\{u_n\}$, for $n > k$ as follows

$$u_n = pu_{n-1} + qu_{n-2} + u_{n-3} + \cdots + u_{n-k}, \tag{3}$$

with nonnegative initials $u_t \geq 0$ for $0 \leq t < k$ and assumed that at least one of them is different from zero.

The author [2] generalized the results given in (2) for the terms of generalized order-k Fibonacci sequence $\{u_n(p,q,2)\}$ as shown : Then there exist a positive integer n_0 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_n(p,q,2)} \right)^{-1} \right\| = u_n(p,q,2) - u_{n-1}(p,q,2), \ (n \geq n_0),$$

where $p \geq q$ and $\|\cdot\|$ denotes the nearest integer. (Clearly $\|x\| = \left\lfloor x + \frac{1}{2} \right\rfloor$).

Recently the authors [3] presented the following results for the order-k recursion $\{u_n(p,1,k)\}$ (with an arbitrary coefficient p and arbitrary k initials but not all of them are zero):

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k(p,1,k)} \right)^{-1} \right\| = u_n(p,1,k) - u_{n-1}(p,1,k), \ (n \geq n_0),$$

$$\left\| \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{u_k(p,1,k)} \right)^{-1} \right\| = (-1)^n \left(u_n(p,1,k) - u_{n-1}(p,1,k) \right), \ (n \geq n_1),$$

and

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_{2k}(p,1,k)} \right)^{-1} \right\| = u_{2n}(p,1,k) - u_{2n-2}(p,1,k), \ (n \geq n_2),$$

where n_0, n_1, n_2 are natural numbers depending on p.

In the rest of this paper, we will obtain generalizations of the results of [3] on the reciprocal sums of order-k recurrence sequence $\{u_n(p,1,k)\}$ mentioned just above. To obtain such generalizations, we will consider the order-k recurrence sequence $\{u_n(p,q,k)\}$ (with two arbitrary coefficients p, q, and arbitrary k initials) instead of the sequence $\{u_n(p,1,k)\}$.

2 Main Results

While considering the order-k sequences defined by (3), we assume that the restriction $p \geq q \geq 1$ throughout this paper.

Our first main result is

Theorem 1 *Let $\{u_n(p,q,k)\}$, briefly $\{u_n\}$, be an order-k sequence defined by (3) with the restriction $p \geq q \geq 1$. Then there exists a positive integer n_0 such that*

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1}, \ (n \geq n_0).$$

Before the proof, we need the following lemmas :

Lemma 2 *Let p and q be positive reals with $p \geq q \geq 1$ and $k \in \mathbb{N}$ with $k \geq 2$. Then for the polynomial*

$$f(x) = x^k - px^{k-1} - qx^{k-2} - x^{k-3} - \cdots - x - 1, \tag{4}$$

we have

(i) $f(x)$ has exactly one positive real root α with $p < \alpha < p + 1$.

(ii) other $k - 1$ roots of $f(x)$ are within the unit circle in the complex plane.

PROOF. Let

$$\begin{aligned} g(x) &= (x-1)\, f(x) \\ &= x^{k+1} - (p+1)\, x^k + (p-q)\, x^{k-1} + (q-1)\, x^{k-2} + 1. \end{aligned}$$

The case $q = 1$ was given in [3]. We will consider two cases $p = q$ and $p > q$.

Case 1: If $p = q$, then

$$g(x) = x^{k+1} - (p+1)\, x^k + (p-1)\, x^{k-2} + 1.$$

This case is very similar to the case $q = 1$ so we omit it here.

Case 2: For $p > q > 1$, we have five nonnegative coefficients in the polynomial $g(x)$ given by

$$g(x) = x^{k+1} - (p+1)\, x^k + (p-q)\, x^{k-1} + (q-1)\, x^{k-2} + 1.$$

According to Descarte's rule, $f(x)$ has at most one positive real root and so $g(x)$ has at most two positive real roots (Clearly one of them is 1).

Now we examine that there exists an another positive real root. Since $p > 1$ and $k \geq 2$ then

$$\begin{aligned} g(p) &= \frac{1}{p^2}\left(p^k q + p^2 - p^k - p^{k+1} q\right) \\ &= \frac{1}{p^2}\left(p^k q\,(1-p) + \left(p^2 - p^k\right)\right) \\ &< 0, \end{aligned}$$

and also since $p^2 > pq$ and $p > 1$ we have

$$
\begin{aligned}
g(p+1) = & \frac{1}{(p+1)^2}\left((p+1)^k\left(p^2-1+p-pq\right)+2p+p^2+1\right) \\
& > 0.
\end{aligned}
$$

Thus there exist an another positive real root α of $g(x)$ with $p < \alpha < p+1$. As a result of this $f(x)$ has exactly one positive real root ($\alpha \in \mathbb{R}$) with $p < \alpha < p+1$. So the proof of lemma (i) is complete.

By considering the lemma (i), we have

$$
\begin{aligned}
&\text{if } x \in \mathbb{R} \text{ such that } x > \alpha, \text{ then } f(x) > 0, \\
&\text{if } x \in \mathbb{R} \text{ such that } 0 < x < a, \text{ then } f(x) < 0,
\end{aligned} \tag{5}
$$

and

$$
\begin{aligned}
&\text{if } x \in \mathbb{R} \text{ such that } x > \alpha, \text{ then } g(x) > 0, \\
&\text{if } x \in \mathbb{R} \text{ such that } 1 < x < a, \text{ then } g(x) < 0.
\end{aligned} \tag{6}
$$

To complete the proof of Lemma (ii), it is sufficient to show that there is no root on and outside of the unit circle.

Claim 1: $f(x)$ has no complex root z_1 with $|z_1| > \alpha$.

Assume that there exists such a root. So we have

$$
f(z_1) = z_1^k - pz_1^{k-1} - qz_1^{k-2} - z_1^{k-3} - \cdots - z_1 - 1 = 0
$$

and then we obtain

$$
\begin{aligned}
\left|z_1^k\right| &\leq p\left|z_1^{k-1}\right| + q\left|z_1^{k-2}\right| + \left|z_1^{k-3}\right| + \cdots + |z_1| + 1 \\
f(|z_1|) &= |z_1|^k - p\,|z_1|^{k-1} - q\,|z_1|^{k-2} - |z_1|^{k-3} - \cdots - |z_1| - 1 \leq 0.
\end{aligned}
$$

This contradicts with (5).

Claim 2: $f(x)$ has no complex root z_2 with $1 < |z_2| < \alpha$.

Suppose that there exists such a root. Since $f(z_2) = 0$,

$$
g(z_2) = z_2^{k+1} - (p+1)\, z_2^k + (p-q)\, z_2^{k-1} + (q-1)\, z_2^{k-2} + 1 = 0,
$$

which implies

$$
(p+1)\,|z_2|^k \leq |z_2|^{k+1} + (p-q)\,|z_2|^{k-1} + (q-1)\,|z_2|^{k-2} + 1.
$$

So we have $g(|z_2|) \geq 0$. But this is a contradiction with (6).

Claim 3: On the circle $|z_3| = \alpha$ and $|z_3| = 1$, $f(x)$ has the unique root α.

Let $z_3 \neq \alpha$ and either $|z_3| = \alpha$ or $|z_3| = 1$ and also $f(z_3) = 0$, then

$$g(z_3) = z_3^{k+1} - (p+1)\, z_3^{k} + (p-q)\, z_3^{k-1} + (q-1)\, z_3^{k-2} + 1 = 0.$$

So we get

$$(p+1)\, |z_3|^{k} \leq |z_3|^{k+1} + (p-q)\, |z_3|^{k-1} + (q-1)\, |z_3|^{k-2} + 1.$$

Since α and 1 are also the roots of $g(z)$,

$$\begin{aligned} &\left| z_3^{k+1} + (p-q)\, z_3^{k-1} + (q-1)\, z_3^{k-2} + 1 \right| \\ &= |z_3|^{k+1} + (p-q)\, |z_3|^{k-1} + (q-1)\, |z_3|^{k-2} + 1. \end{aligned}$$

The equality holds if and only if all parts lie on the same ray issuing from the origin. One of the parts is 1 (see [4]). So the other parts, z_3^{k+1}, $(p-q)\, z_3^{k-1}$, $(q-1)\, z_3^{k-2}$, must be element of $\mathbb{R}^+$. Since $(p-q), (q-1) \in \mathbb{R}^+$, z_3^{k+1}, z_3^{k-1} and z_3^{k-2} must be elements of $\mathbb{R}^+$. Therefore we obtain $z_3 \in \mathbb{R}^+$. There are two possibilities $z_3 = 1$ or $z_3 = \alpha$. Since $f(1) \neq 0$ the case $z_3 = 1$ is ruled out. From lemma (i) we know that $f(x)$ has exact one positive real root α. So the case $z_3 = \alpha$ has already known. Since multiple roots are counted separately by Descarte's rule, there is not an another positive real root. From these tree claims, lemma (ii) is proven.

Consequently, $f(x)$ has exactly one positive real root α with $p < \alpha < p+1$ and the other roots are within the unit circle.

Lemma 3 *Let $k \geq 2$, then the closed formula of $\{u_n\}$ is given by*

$$u_n = a\alpha^n + O(c^{-n}) \qquad (n \to \infty),$$

where $a > 0$, $c > 1$ and α is the positive real root of (4).

PROOF. Let $\alpha, \alpha_1, \alpha_2, ..., \alpha_t$ with $|\alpha_i| < 1$ for $1 \leq i \leq k-1$ be distinct roots of $f(x)$ and r_j for $j = 1, 2, ..., t$ denotes the multiplicity of the root α_j. Then u_n can be written as follows

$$u_n = a\alpha^n + \sum_{i=1}^{t} P_i(n) \alpha_i^n,$$

where $P_i(n) \in \mathbb{R}[x]$ with $\deg P_i = r_i - 1$, $r_1 + r_2 + \cdots + r_t = k-1$ and $a \in \mathbb{R}^+$. Since $|\alpha_i| < 1$ for $1 \leq i \leq t$, each term of tail goes to 0 as $n \to \infty$. So we can find constant $K \in \mathbb{R}$ and $c \in \mathbb{R}$ with $c > 1$ for $n > n_0$ such that

$$\sum_{i=1}^{t} P_i(n)\alpha_i^n \leq Kc^{-n},$$

which completes the proof. (Note that if all roots of $f(x)$ are distinct we can choose $c^{-1} = \max\{|\alpha_1|, |\alpha_2|, ..., |\alpha_{k-1}|\}$ and $K = k - 1$.)

PROOF. [Proof of Theorem 1] From the geometric series as $\epsilon \to 0$ we have

$$\frac{1}{1 \pm \epsilon} = 1 \pm \epsilon + O(\epsilon^2) = 1 + O(\epsilon).$$

Using Lemma 2, we have

$$\begin{aligned} \frac{1}{u_k} &= \frac{1}{a\alpha^k + O(c^{-k})} = \frac{1}{a\alpha^k\left(1 + O((\alpha c)^{-k})\right)} \\ &= \frac{1}{a\alpha^k}\left(1 + O((\alpha c)^{-k})\right) = \frac{1}{a\alpha^k} + O\left(\left(\alpha^2 c\right)^{-k}\right). \end{aligned}$$

Thus

$$\begin{aligned} \sum_{k=n}^{\infty} \frac{1}{u_k} &= \frac{1}{a}\sum_{k=n}^{\infty} \frac{1}{\alpha^k} + O\left(\sum_{k=n}^{\infty} \left(\alpha^2 c\right)^{-k}\right) \\ &= \frac{\alpha}{a(\alpha - 1)}\alpha^{-n} + O\left(\left(\alpha^2 c\right)^{-n}\right). \end{aligned}$$

By taking reciprocal we get

$$\begin{aligned} \left(\sum_{k=n}^{\infty} \frac{1}{u_k}\right)^{-1} &= \frac{1}{\frac{\alpha}{a(\alpha-1)}\alpha^{-n} + O\left((\alpha^2 c)^{-n}\right)} \\ &= \frac{\alpha - 1}{\alpha} a\alpha^n \left(1 + O\left((\alpha c)^{-n}\right)\right) \\ &= \frac{\alpha - 1}{\alpha} a\alpha^n + O\left(c^{-n}\right) \\ &= u_n - u_{n-1} + O\left(c^{-n}\right). \end{aligned}$$

So there there exists n_0 such that the last error term becomes less than $1/2$ which completes the proof.

Theorem 4 *Let* $\{u_n(p, q, k)\}$, *briefly* $\{u_n\}$, *be an order-k sequence defined by (3) with a restriction* $p \geq q$. *Then there exists a positive integer* n_1, *such* *that*

$$\left\| \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{u_k} \right)^{-1} \right\| = (-1)^n \left(u_n + u_{n-1} \right), \ (n \geq n_1).$$

PROOF. Here we start to the proof with computing the summand term

$$\frac{(-1)^k}{u_k} = \frac{(-1)^k}{a\alpha^k + O(c^{-k})} = \frac{(-1)^k}{a\alpha^k} \left(1 + O\left((\alpha c)^{-k} \right) \right).$$

Then we have

$$\sum_{k=n}^{\infty} \frac{(-1)^k}{a\alpha^k} \left(1 + O\left((\alpha c)^{-k} \right) \right) = \frac{\alpha}{a(-\alpha)^n (\alpha + 1)} + O\left(\left(\alpha^2 c \right)^{-n} \right).$$

By taking reciprocal,

$$\begin{aligned} \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{u_k} \right)^{-1} &= \frac{a(-\alpha)^n (\alpha+1)}{\alpha} \left(1 + O\left((\alpha c)^{-n} \right) \right) \\ &= (-1)^n \left(a\alpha^n + a\alpha^{n-1} \right) + O\left(c^{-n} \right) \\ &= (-1)^n \left(u_n + u_{n-1} \right) + O\left(c^{-n} \right). \end{aligned}$$

Then we can find integer n_1 such that the error term is less than $1/2$ for $n \geq n_1$.

The following result could be proven similar to the previous results.

Theorem 5 *For the sequence which defined in (3) with a restriction $p \geq q$. Then there exist positive integers n_2 and n_3 such that*

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_{tk+r}} \right)^{-1} \right\| = \left(u_{tn+r} - u_{tn-t+r} \right), \ (n \geq n_2),$$

$$\left\| \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{u_{tk+r}} \right)^{-1} \right\| = (-1)^n \left(u_{tn+r} + u_{tn-t+r} \right), \ (n \geq n_3),$$

where t and r positive integers with $0 \leq r < t$.

Now we present some examples of our results. When $q = 1$, $t = 2$, $r = 0$ and $r = 1$ in the previous theorem, respectively, we get same results given in [3].

When we take $p = 2$, $q = 1$, $k = 2$, with initials $u_0 = 0$ and $u_1 = 1$, respectively, we have same result in [6]. In addition we have more results such as

$$\left\| \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{P_k} \right)^{-1} \right\| = (-1)^n \left(P_n + P_{n-1} \right), \ (n \geq 1), \tag{7}$$

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{P_{tk+r}} \right)^{-1} \right\| = \ \left(P_{tn+r} - P_{tn-t+r} \right), (n \geq n_0).$$

Identity (7) can be also found in [3].

When $p = q = 1$, $k = 2$, $t = 5$ and $r = 3$ with initials $u_0 = 0$ and $u_1 = 1$, we obtain new result as follows,

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{F_{5k+3}} \right)^{-1} \right\| = \left(F_{5n+3} - F_{5n-2} \right), \ (n \geq 1),$$

$$\left\| \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{F_{5k+3}} \right)^{-1} \right\| = (-1)^n \left(F_{5n+3} + F_{5n-2} \right), \ (n \geq 1).$$

For example, we consider the sequence $\{u_n\}$ defined for $n > 3$ by

$$u_n = 7u_{n-1} + 4u_{n-2} + u_{n-3} + u_{n-4},$$

with initials $u_0 = 0$, $u_1 = 1$, $u_2 = 2$ and $u_3 = 3$. Then, by Theorems 1 and 5, we obtain

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1}, \ (n \geq n_0),$$

$$\left\| \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{u_{tk+r}} \right)^{-1} \right\| = (-1)^n \left(u_{tn+r} + u_{tn-t+r} \right), \ (n \geq n_1),$$

where n_0 and n_1 are determined according to the initial values and the roots of characteristic equation of sequence $\{u_n\}$.

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