A CURIOUS MATRIX-SUM IDENTITY AND CERTAIN FINITE SUMS IDENTITIES

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ABSTRACT. In this paper, we consider two generalized binary sequences and then give a generalization of a matrix equality proposed as an advanced problem. Then we derive new certain finite sums including the generalized binary sequences as applications.

1. INTRODUCTION

Define the generalized Fibonacci and Lucas sequences as for $n > 0$

$$\begin{aligned} U_{n+1} &= PU_n + U_{n-1}, \\ V_{n+1} &= PV_n + V_{n-1} \end{aligned}$$

where $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = P$, respectively. When $P = 1$, then $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number).

The Binet formulæ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n,$$

where α and β are the roots of $x^2 - Px - 1 = 0$.

From [8], it is known that for $k \geq 0$ and $n > 1$

$$\begin{aligned} U_{kn} &= V_k U_{k(n-1)} + (-1)^{k+1} U_{k(n-2)}, \\ V_{kn} &= V_k V_{k(n-1)} + (-1)^{k+1} V_{k(n-2)}, \end{aligned}$$

Define the matrix A as follows

$$A = \begin{bmatrix} V_k & (-1)^{k+1} \\ 1 & 0 \end{bmatrix}. \tag{1.1}$$

In [3], the authors obtained new results for the sequence $\{U_{kn}\}$ by using its generating matrix. For example, we have

$$A^n = \frac{1}{U_k} \begin{bmatrix} U_{k(n+1)} & (-1)^{k+1} U_{kn} \\ U_{kn} & (-1)^{k+1} U_{k(n-1)} \end{bmatrix} \tag{1.2}$$

and

$$A^{-n} = \frac{1}{(-1)^{kn} U_k} \begin{bmatrix} (-1)^{k+1} U_{k(n-1)} & (-1)^k U_{kn} \\ -U_{kn} & U_{k(n+1)} \end{bmatrix}. \tag{1.3}$$

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If $\alpha(k)$ and $\beta(k)$ are the roots of equation $x^2 - V_k x + (-1)^k = 0$. Then the Binet formulas of the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$ are

$$U_{kn} = U_k \frac{\alpha(k)^n - \beta(k)^n}{\alpha(k) - \beta(k)} \text{ and } V_{kn} = \alpha(k)^n + \beta(k)^n .$$

Note that $\alpha(1) = \alpha$ and $\beta(1) = \beta$, where α and β are defined as before.

Define the matrices G and I as shown

$$G = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In [4], as an advanced problem, it was proposed that for $m > 0$, prove that

$$\sum_{n=0}^{m-1} 2^n G^{2^n} \left(G^{2^n} + I^{2^n} \right)^{-1} = e_{2^m} E_{2^m} - (G + I),$$

where

$$e_m = m / (F_{m+1} + F_{m-1} - 2) \text{ and } E_m = \begin{bmatrix} F_{m+1} - 1 & F_m \\ F_m & F_{m-1} - 1 \end{bmatrix}$$

and F_m is the mth Fibonacci number. Also note that

$$G^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

The problem was solved by Bruckman, Kappus and Seiffert and the proposer (see [1]).

Many authors have used matrix methods to derive some identities, combinatorial representations for terms of linear recurrence relations (for more details see [2, 5, 6, 7, 3, 9]). Matrix methods have been useful tools for solving problems stemming from number theory and deriving combinatorial identities for certain linear recurrences.

Here we recall a well known formula for later use: Let M and N be two matrices with the same order. The Waring's formula is

$$M^n + N^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (MN)^j (M+N)^{n-2j} \ , \quad (n > 0) . \tag{1.4}$$

In this paper, motivated by the advanced problem, we will consider general Fibonacci and Lucas numbers, and, related generating matrices, then we derive more general case of the problem. As consequences of our generalization, we obtain several finite sums for the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$. We also derive new combinatorial identites for the sequences.

2. A Generalization of the Problem

In this section, we present a generalization of the problem mentioned in the Introduction. Define 2×2 matrices H and R in terms of the matrices A, A^{-1} and their nth powers as follows:

$$H = A + A^{-1} = \left(1 - (-1)^k \right) A + (-1)^k V_k I \tag{2.1}$$

and

$$R_n = A^n + A^{-n}.$$

Here clearly $H=R_1$. If we sum the the equations (1.2)-(1.3), we get

$$R_n=\begin{cases} V_{kn}I & \text{if } n \text{ is even,}\\ \left(U_{kn}/U_k\right)H & \text{if } n \text{ is odd and } k \text{ is odd,}\\ V_{kn}I & \text{if } n \text{ is odd and } k \text{ is even.}\end{cases} \tag{2.2}$$

Since the eigenvalues of H are λ_1 and λ_2, we obtain

$$\begin{aligned} H^n &= \begin{cases} \lambda_t^n I & \text{if } n \text{ is even,}\\ \lambda_t^{n-1}H & \text{if } n \text{ is odd,}\end{cases} \\ H^{-n} &= H^n\lambda_t^{-2n} \text{ for } t=1,2. \end{aligned} \tag{2.3}$$

Theorem 1. *For $m\geq 1$ and odd $k\geq 1$,*

$$\sum_{n=0}^{m-1} 2^n A^{2^n}\left(A^{2^n}+I^{2^n}\right)^{-1}=c_{2^m}C_{2^m}-\frac{1}{V_k}\left(A+I\right), \tag{2.4}$$

where

$$c_m=\frac{m}{U_k\left(V_{km}-2\right)},\ C_m=\begin{bmatrix} U_{k(m+1)}-U_k & U_{km}\\ U_{km} & U_{k(m-1)}-U_k\end{bmatrix}$$

and I is the unit matrix of order two.

Proof. Denote S_m be the sum on the LHS of (2.4). Let $W\left(n\right)=nA^n\left(A^n+I\right)^{-1}$. Since $\det\left(A^n+I\right)=V_{kn}+\left(-1\right)^{kn}+1$, we write

$$\left(A^n+I\right)^{-1}=\frac{1}{U_k\left[V_{kn}+\left(-1\right)^{kn}+1\right]}\begin{bmatrix} U_{k(n-1)}+U_k & -U_{kn}\\ -U_{kn} & U_{k(n+1)}+U_k\end{bmatrix}$$

and

$$\begin{aligned} W\left(n\right)&=nA^n\left(A^n+I\right)^{-1}\\ &=\frac{n}{U_k\left[V_{kn}+\left(-1\right)^{kn}+1\right]}\begin{bmatrix} U_{k(n+1)}+\left(-1\right)^{kn}U_k & U_{kn}\\ U_{kn} & U_{k(n-1)}+\left(-1\right)^{kn}U_k\end{bmatrix}, \end{aligned}$$

where

$$W\left(1\right)=\frac{1}{V_k}\begin{bmatrix} V_k-1 & 1\\ 1 & -1\end{bmatrix}.$$

Also

$$\begin{aligned} &c_2C_2-\frac{1}{V_k}\left(A+I\right)\\ &=\frac{2}{U_k\left(V_{2k}-2\right)}\begin{bmatrix} U_{3k}-U_k & U_{2k}\\ U_{2k} & 0\end{bmatrix}-\frac{1}{V_k}\begin{bmatrix} V_k+1 & 1\\ 1 & 1\end{bmatrix}\\ &=\frac{1}{V_k}\begin{bmatrix} V_k-1 & 1\\ 1 & -1\end{bmatrix}=S_1. \end{aligned}$$

Thus, the equality in (2.4) for $m=1$ is valid. Let N denote the set of positive integers m satisfied the equation (2.4). We have shown $1\in N$. Suppose that $m\in N$.

Taking $u = 2^m$ and by the inductive hypothesis, we get

$$\begin{aligned} S_{m+1} &= S_m + W(u) = c_u C_u - \frac{1}{V_k}(A+I) + \\ & \frac{u}{U_k\left[V_{ku}+(-1)^{ku}+1\right]}\begin{bmatrix} U_{k(u+1)}+(-1)^{ku}U_k & U_{ku} \\ U_{ku} & U_{k(u-1)}+(-1)^{ku}U_k \end{bmatrix} \\ &= \frac{u}{U_k\left[V_{ku}+(-1)^{ku}-1\right]}\begin{bmatrix} U_{k(u+1)}-(-1)^{ku}U_k & U_{ku} \\ U_{ku} & U_{k(u-1)}-(-1)^{ku}U_k \end{bmatrix} + \\ & \frac{u}{U_k\left[V_{ku}+(-1)^{ku}+1\right]}\begin{bmatrix} U_{k(u+1)}+(-1)^{ku}U_k & U_{ku} \\ U_{ku} & U_{k(u-1)}+(-1)^{ku}U_k \end{bmatrix} \\ & -\frac{1}{V_k}(A+I), \end{aligned}$$

which after some arrangements equals

$$\begin{aligned} &= \frac{u}{(V_{2ku}-2)}\begin{bmatrix} 2\left(U_{k(2u+1)}-U_k\right) & 2U_{2ku} \\ 2U_{2ku} & -2\left(-U_{k(2u-1)}+U_k\right) \end{bmatrix} - \frac{1}{V_k}(A+I) \\ &= c_{2u}\begin{bmatrix} U_{k(2u+1)}-U_k & U_{2ku} \\ U_{2ku} & U_{k(2u-1)}-U_k \end{bmatrix} - \frac{1}{V_k}(A+I) \\ &= c_u C_u - \frac{1}{V_k}(A+I), \end{aligned}$$

as claimed. □

Corollary 1. *For $n \geq 1$ and odd integer $k \geq 1$*

$$\sum_{t=0}^{n-1} \frac{2^t\left(U_{k(2^t+1)}+U_k\right)}{U_k\left(V_{k2^t}+(-1)^{k2^t}+1\right)} = \frac{2^n\left(U_{k(2^n+1)}-U_k\right)}{U_k\left(V_{k2^n}-2\right)} - \frac{V_k-1}{V_k}$$

and

$$\sum_{t=0}^{n-1} \frac{2^t U_{k2^t}}{U_k\left(V_{k2^t}+(-1)^{k2^t}+1\right)} = \frac{2^n U_{k2^n}}{U_k\left(V_{k2^n}-2\right)} - \frac{1}{V_k}.$$

Proof. The proof follows by equating the entries $(1,1)$ and $(1,2)$ of the matrix equations in (2.4). □

3. Some Miscellaneous Formulæ

In this section we consider the matrix A in (1.1) and then use some known matrix methods given in [3] to derive some new sums formulæ for the sequences $\{U_{kn}\}$ and $\{V_{kn}\}$.

Theorem 2. *For $n \geq 0$ and $k > 0$,*

$$\sum_{j=0}^{n} \binom{n}{j}\left(\frac{-1}{V_k}\right)^j U_{kj} = -\frac{U_{kn}}{V_k^n}, \tag{3.1}$$

$$\sum_{j=0}^{n-1} \binom{n}{j}\left(\frac{-1}{V_k}\right)^j U_{kj} = \begin{cases} -2U_{kn}/V_k^n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{3.2}$$

Proof. By the characteristic polynomial of matrix A, we write

$$\begin{aligned} A^n &= \left(-(-1)^k A^{-1} + V_k I\right)^n \\ &= V_k^n \sum_{j=0}^{n} \binom{n}{j} (-1)^{j(k+1)} (V_k A)^{-1}. \end{aligned} \tag{3.3}$$

Equating $(2,1)$-entries of (3.3) gives the claimed results. □

Theorem 3. *For $n \geq 0$, odd $k > 0$ and $t = 1, 2$,*

$$\sum_{j=0}^{n} \binom{n}{j} \left(\frac{-2}{V_k}\right)^j U_{kj} = \begin{cases} (-2)\,\lambda_t^{n-1} \dfrac{U_k}{V_k^n} & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases}$$

Proof. Using (2.1) and (2.2), we get

$$\begin{aligned} H^n &= \left(\left(1-(-1)^k\right) A + V_k (-1)^k I\right)^n \\ &= \frac{V_k^n}{(-1)^{kn}} \sum_{j=0}^{n} \binom{n}{j} \left(\frac{(-1)^k - 1}{V_k}\right)^j A^j \\ &= \frac{V_k^n}{(-1)^{kn}} \sum_{j=0}^{n} \binom{n}{j} \left(\frac{-2}{V_k}\right)^j A^j. \end{aligned} \tag{3.4}$$

From the matrix equality in (3.4) and using (2.3), we have the conclusion. □

Theorem 4. *For any integer k and even n,*

$$\begin{aligned} &\sum_{j=0}^{(t-2)/2} \binom{t}{j} V_{kn(t-2j)} = V_{kn}^t - \binom{t}{t/2} && \text{if } t \text{ is even}, \\ &\sum_{j=0}^{(t-1)/2} \binom{t}{j} V_{kn(t-2j)} = V_{kn}^t && \text{if } t \text{ is odd}, \end{aligned}$$

and for $m = 1, 2$, odd integers k and n,

$$\begin{aligned} &\sum_{j=0}^{(t-2)/2} \binom{t}{j} V_{kn(t-2j)} = \left(\frac{U_{kn}}{U_k}\right)^t \lambda_m^t - \binom{t}{t/2} && \text{if } t \text{ is even}, \\ &\sum_{j=0}^{(t-1)/2} \binom{t}{j} \frac{U_{kn(t-2j)}}{U_k} = \left(\frac{U_{kn}}{U_k}\right)^t \lambda_m^{t-1} && \text{if } t \text{ is odd}. \end{aligned}$$

Proof. From (2.2), we have

$$R_n^t = \sum_{j=0}^{t} \binom{t}{j} A^{n(2j-t)}.$$

After some arrangements, we write

$$R_n^t = \begin{cases} \dbinom{t}{t/2} I + \displaystyle\sum_{j=0}^{(t-2)/2} \binom{t}{j} R_{n(t-2j)} & \text{if } t \text{ is even}, \\ \displaystyle\sum_{j=0}^{(t-1)/2} \binom{t}{j} R_{n(t-2j)} & \text{if } t \text{ is odd}. \end{cases}$$

By equating corresponding entries of the matrix equation above, the proof follows. □

Theorem 5. *For $n > 0$ and even k,*

$$\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{j} V_{k(n-2j)} = \begin{cases} V_k^n - \binom{n}{n/2} & \text{if } n \text{ is even,} \\ V_k^n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Since $V_k I = A^{-1} + A$, we write

$$V_k^n I = \sum_{j=0}^{n} \binom{n}{j} A^{2j-n} = \begin{cases} \binom{n}{n/2} I + \sum\limits_{j=0}^{(n-2)/2} \binom{n}{j} R_{n-2j} & \text{if } n \text{ is even,} \\ \sum\limits_{j=0}^{(n-1)/2} \binom{n}{j} R_{n-2j} & \text{if } n \text{ is odd,} \end{cases}$$

which, by equating the corresponding entries and using (1.3),(2.1) and (2.2), gives us the claimed result. □

Theorem 6. *For $k > 0$, $t = 1, 2$,*

$$\sum_{j=0}^{n} \binom{n}{j} U_{2kj} = \begin{cases} \lambda_t^n U_{kn} & \text{if } n \text{ is even,} \\ \lambda_t^{n-1} (U_{k(n+1)} + U_{k(n-1)}) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. From (1.2) and (2.1), we get

$$(HA)^n = \begin{cases} \lambda_t^n A^n & \text{if } n \text{ is even,} \\ \lambda_t^{n-1} H A^n & \text{if } n \text{ is odd.} \end{cases} \tag{3.5}$$

From (2.1), we have $HA = A^2 + I$ and so

$$(HA)^n = \left(A^2 + I\right)^n = \sum_{j=0}^{n} \binom{n}{j} A^{2j}. \tag{3.6}$$

Equating $(2,1)$-entries of (3.6) gives the conclusion. □

Theorem 7. *For $t = 1, 2$,*

$$\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} \left(\lambda_t^2\right)^{\lfloor n/2 \rfloor - j} = \begin{cases} V_{kn} & \text{if } n \text{ is even,} \\ U_{kn}/U_k & \text{if } n \text{ and } k \text{ are odd,} \end{cases} \tag{3.7}$$

and for odd n and even k,

$$\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} \left(\lambda_t^2\right)^{\lfloor n/2 \rfloor - j} V_k = V_{kn}.$$

Proof. By (1.4) and (2.1), we write that for $n > 0$

$$R_n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} H^{n-2j}. \tag{3.8}$$

By equating $(1,1)$-entries of (3.8) and by (2.2) and (3.3), the claim is proven. □

Theorem 8.

$$U_{k(2n+1)}+U_k=\begin{cases} U_{k(n+1)}V_{kn} & \text{if } n \text{ is even or if } n \text{ is odd and } k \text{ is even,}\\ U_{kn}V_{k(n+1)} & \text{if } n \text{ and } k \text{ are odd.}\end{cases}$$

Proof. By (3.8), we have

$$\begin{aligned} A^{2n}+I &= A^n\left(A^n+A^{-n}\right)=A^nR_n \\ &= \sum_{j=0}^{\lfloor n/2\rfloor}(-1)^j\frac{n}{n-j}\binom{n-j}{j}A^nH^{n-2j}. \end{aligned} \tag{3.9}$$

The proof follows by equating $(1,1)$-entries of (3.9) and from (3.7). □

The matrix equation

$$M^n+N^n=(M+N)\sum_{j=1}^{n}(-1)^{j-1}M^{n-j}N^{j-1} \tag{3.10}$$

is valid if $MN=NM$ and n is odd.

The matrix equation

$$\sum_{j=0}^{n}M^j=\left(M^{n+1}-I\right)(M-I)^{-1} \tag{3.11}$$

is valid if all eigenvalues of M are different from 1.

Theorem 9. *For positive odd integers n and k,*

$$\frac{U_{kn}}{U_k}=\sum_{j=1}^{(n-1)/2}(-1)^{j-1}V_{k(n-2j+1)}+(-1)^{(n-1)/2}.$$

Proof. By (3.10), we write

$$\begin{aligned} R_n &= M^n+M^{-n}=\left(M+M^{-1}\right)\sum_{j=1}^{n}(-1)^{j-1}M^{n-2j+1} \\ &= H\sum_{j=1}^{n}(-1)^{j-1}M^{n-2j+1}=H\left((-1)^{(n-1)/2}I+\sum_{j=1}^{(n-1)/2}(-1)^{j-1}R_{n-2j+1}\right). \end{aligned}$$

From (2.2), we have that for odd n,

$$R_n=(-1)^{(n-1)/2}H+H\sum_{j=1}^{(n-1)/2}(-1)^{j-1}V_{k(n-2j+1)}I. \tag{3.12}$$

The proof follows by equating $(1,1)$-entries of (3.12). □

Theorem 10. *For odd n and any integer k,*

$$\sum_{j=0}^{(n-1)/2}\lambda_t^{2j}\left(U_{k(2j+3)}+2U_{k(2j+1)}\right)=\lambda_t^{n+1}U_{kn}+(-1)^kU_k,\ t=1,2.$$

Proof. By (3.11),(2.1) and (3.5), we write for odd n,

$$\sum_{j=0}^{n}(HA)^j = \left(H^{n+1}A^{n+1}-I\right)(HA-I)^{-1} = \left(H^{n+1}A^{n+1}-I\right)A^{-2} = \left(\lambda_t^{n+1}A^{n+1}-I\right)A^{-2}. \tag{3.13}$$

Again, by (2.1) and (3.5), the LHS of (3.13) can be rewritten as

$$\sum_{j=0}^{n}(HA)^j = \sum_{j=0}^{(n-1)/2}\lambda_t^{2j}\left(A^{2j}+HA^{2j+1}\right) = \sum_{j=0}^{(n-1)/2}\lambda_t^{2j}\left(A^{2j+2}+2A^{2j}\right),\ t=1,2. \tag{3.14}$$

Equating $(1,1)$-entries of RHS of (3.13) and (3.14), the proof follows by (1.3). □

4. Some Congruence Properties

Now we present some congruence properties of $\{U_{kn}\}$ and $\{V_{kn}\}$ by using (3.7). If $p>2$ is a prime, then from (3.7), we write for $t=1,2$,

$$\frac{U_{kp}}{U_k} = \lambda_t^{p-1} + \sum_{j=1}^{(p-1)/2}(-1)^j\frac{p}{p-j}\binom{p-j}{j}\lambda_t^{p-1-2j}. \tag{4.1}$$

Note that the sum on the RHS of (4.1) is divisible by p and we write

$$\frac{U_{kp}}{U_k} \equiv \lambda_t^{p-1} \pmod{p}.$$

Equivalently, we can state that $(U_{kp}/U_k) \equiv 0 \pmod p$ if $\lambda_t^2 \equiv 0 \pmod p$ while $(U_{kp}/U_k) \equiv 1$ (or -1) $\pmod p$ if λ_t^2 is (is not) a quadratic residue modulo p.

For positive odd integer n, we rewrite (4.1) as for odd integers n and k,

$$\frac{U_{kn}}{U_k} = (-1)^{\frac{n-1}{2}}\frac{2n}{n+1}\binom{(n+1)/2}{(n-1)/2} + \sum_{j=0}^{(n-3)/2}(-1)^j\frac{n}{n-j}\binom{n-j}{j}\lambda_t^{n-1-2j} = n(-1)^{\frac{n-1}{2}} + \sum_{j=0}^{(n-3)/2}(-1)^j\frac{n}{n-j}\binom{n-j}{j}\lambda_t^{n-1-2j}. \tag{4.2}$$

From (4.2), the following congruence is derived for odd integers n and k,

$$\frac{U_{kn}}{U_k} \equiv n(-1)^{\frac{n-1}{2}} \pmod{\lambda_t^2}.$$

Using the same procedure, for even integer n and any integer k, we get

$$V_{kn} \equiv 2(-1)^{\frac{n}{2}} \pmod{\lambda_t^2}.$$

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