PARTIAL SUMS OF THE GAUSSIAN q-BINOMIAL COEFFICIENTS, THEIR RECIPROCALS, SQUARE AND SQUARED RECIPROCALS WITH APPLICATIONS

EMRAH KILIÇ AND ILKER AKKUS

ABSTRACT. In this paper, we shall derive formulæ for *partial* sums of the Gaussian q-binomial coefficients, their reciprocals, squares and squared reciprocals. To prove the claimed results, we use q-calculus. As applications of our results, we give some interesting generalized Fibonomial sums formulæ.

1. Introduction

For $n > 1$, define the second order linear sequences $\{U_n\}$ and $\{V_n\}$ by

$$U_n = pU_{n-1} + U_{n-2}\ ,\quad U_0 = 0\ ,\quad U_1 = 1,$$
$$V_n = pV_{n-1} + V_{n-2}\ ,\quad V_0 = 2\ ,\quad V_1 = p.$$

When $p = 1$, $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number), resp. Falcon and Plaza named the previous sequences as k-Fibonacci and k-Lucas numbers, see [5,6].

For $n \geq k \geq 1$, define the generalized Fibonomial coefficients by

$$\left\{ {n \atop k} \right\}_U := \frac{U_1 U_2 \ldots U_n}{(U_1 U_2 \ldots U_k)(U_1 U_2 \ldots U_{n-k})}$$

with $\left\{ {n \atop 0} \right\}_U = \left\{ {n \atop n} \right\}_U = 1$. When $p = 1$, we obtain the usual Fibonomial coefficients, denoted by $\left\{ {n \atop k} \right\}_F$. For more details about the Fibonomial and generalized Fibonomial coefficients, see [7,9,22].

The Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = \left(p \pm \sqrt{p^2 + 4}\right)/2$.

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Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_U = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \text{with} \quad q = -\alpha^{-2}.$$

By taking $q = \beta/\alpha$, the Binet formulæ are reduced to the following forms:

$$U_n = \alpha^{n-1} \frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n (1+q^n),$$

where $\mathbf{i} = \sqrt{-1} = \alpha\sqrt{q}$. For later use note that q-form of the coefficient p in the recurrence relations of $\{U_n\}$ and $\{V_n\}$ is $(1+q)(-q)^{-1/2}$.

The Fibonomial coefficients surprisingly appear in several places in the literature (for more details, we refer to [4, 10, 12–14]). Nowadays interesting sums including the Fibonomial coefficients with certain factors or sign functions have been introduced and computed by several authors (see [11, 15–21, 23]).

Marques and Trojovsky [19] presented some Fibonomial sums formulæ with the Fibonacci and Lucas numbers as coefficients. For example, for positive integers m and n, they showed that

$$\sum_{j=0}^{4m+2} (-1)^{\frac{j(j-1)}{2}} \begin{Bmatrix} 4m \\ j \end{Bmatrix}_F F_{n+4m-j} = \frac{1}{2} F_{2m+n} \sum_{j=0}^{4m} (-1)^{\frac{j(j-1)}{2}} \begin{Bmatrix} 4m \\ j \end{Bmatrix}_F L_{2m-j}.$$

Kılıç and Prodinger [16] gave a systematic approach to compute certain sums of squares of Fibonomial coefficients with finite products of generalized Fibonacci and Lucas numbers as coefficients. For example, if n is nonnegative integer, then they proved the following Gaussian q-binomial sums identity

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 (-1)^k q^{k^2-2kn-3k} (1-q^{2k})^2$$
$$= 2(-1)^{n+1} q^{-n^2-2n-2} \frac{(1+q)(1-q^{2n+1})(1-q^{2n+1})}{(1+q^{2n})} \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_{q^2}.$$

Recently Marques and Trojovsky [20] derived various interesting Fibonomial sums formulae with certain weight functions. For example, they gave that for any nonnegative integers l and n,

$$\sum_{j=0}^{4l+3} sgn(2l+1-j) \begin{Bmatrix} 4l+3 \\ j \end{Bmatrix}_F F_{n-j} = \frac{F_{2l}}{F_{4l+3}} \begin{Bmatrix} 4l+3 \\ 2l+1 \end{Bmatrix}_F F_{n-4l-3} \tag{1.1}$$

and

$$\sum_{j=0}^{4l+1} sgn(2l-j) \left\{ \begin{matrix} 4l+1 \\ j \end{matrix} \right\}_F F_{n-j} = -\frac{F_{2l-1}}{F_{4l+1}} \left\{ \begin{matrix} 4l+1 \\ 2l \end{matrix} \right\}_F F_{n-4l-1}, \tag{1.2}$$

where $sgn(x)$ denotes the sign function of x, defined by

$$sgn\,(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Much recently Kılıç and Akkuş [1] generalized all the results of [20] through the Gaussian q-binomial coefficients instead of the Fibonomial coefficients with additional parameters. They also gave analogue of the sums formulae whose upper bounds are even integers. The authors proved their claimed results by mainly and analytically q-calculus, and the celebrated Zeilberger algorithm for some steps of their proofs. For convenience of the readers, we recall two sums formulae from [1]:

$$\sum_{j=0}^{4l+3} sgn(2l+1-j) \begin{bmatrix} 4l+3 \\ j \end{bmatrix}_q (-1)^{\frac{1}{2}j(j-2)} q^{-\frac{1}{2}j(4l-j+2)} (1-q^{n-j}) z^{[2\nmid j]}$$
$$= q^{2-2l^2} \frac{1-q^{2l}}{1-q^{4l+3}} \begin{bmatrix} 4l+3 \\ 2l+1 \end{bmatrix}_q (1-q^{n-4l-3}),$$

and

$$\sum_{j=0}^{4l+1} sgn(2l-j) \begin{bmatrix} 4l+1 \\ j \end{bmatrix}_q (-1)^{\frac{1}{2}j^2} q^{\frac{1}{2}j(j-4l)} (1-q^{n-j}) z^{[2|j]}$$
$$= -(-q)^{\frac{4l+3-4l^2}{2}} \frac{1-q^{2l-1}}{1-q^{4l+1}} \begin{bmatrix} 4l+1 \\ 2l \end{bmatrix}_q (1-q^{n-4l-1}),$$

where $z = (1+q)(-q)^{-1/2}$, resp. If one take $q = \left(p - \sqrt{p^2+4}\right) / \left(p + \sqrt{p^2+4}\right)$, then these sums are reduced to the following generalized Fibonomial sums formulæ: For nonnegative integers l and n,

$$\sum_{j=0}^{4l+3} sgn(2l+1-j) p^{[2\nmid j]} \left\{ \begin{matrix} 4l+3 \\ j \end{matrix} \right\}_U U_{n-j} = \frac{U_{2l}}{U_{4l+3}} \left\{ \begin{matrix} 4l+3 \\ 2l+1 \end{matrix} \right\}_U U_{n-4l-3},$$

and

$$\sum_{j=0}^{4l+1} sgn(2l-j) p^{[2|j]} \left\{ \begin{matrix} 4l+1 \\ j \end{matrix} \right\}_U U_{n-j} = -\frac{U_{2l-1}}{U_{4l+1}} \left\{ \begin{matrix} 4l+1 \\ 2l \end{matrix} \right\}_U U_{n-4l-1},$$

where [ ] stands for the Iverson notation (see [8]). We would like to take attention of the readers to factors $p^{[2|j]}$ and $p^{[2\nmid j]}$ in these generalizations just above. These are not easily seen while deriving the generalized sums formulæ. Indeed, when $p = 1$, these generalized Fibonomial sums formulæ are reduced to the sums formulae (1.1) and (1.2) given in [20]. Similarly when $q = \left(1+\sqrt{5}\right)/\left(1-\sqrt{5}\right)$ or equivalently $p = 1$, some of the results of [1] cover the results of [20].

Othsuka conjectured the following two advanced problems, advanced problem H-764 and H-768 (see [2,3]). Advanced Problem H-764: For $n \geq 1$, prove that

$$(i)\ \sum_{k=0}^{n} F_{2(n-k)} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_F = \frac{F_n F_{n+1}}{F_{2n-1}} \begin{Bmatrix} 2n \\ n \end{Bmatrix}_F,$$

$$(ii)\ \sum_{k=0}^{n} F_{2(n-k)} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_F^2 = \frac{F_n}{L_n} \begin{Bmatrix} 2n \\ n \end{Bmatrix}_F^2.$$

Advanced Problem H-768: For $n \geq 1$, prove that

$$(i)\ \sum_{k=0}^{n} F_{2(n-k)} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_F^{-1} = \frac{F_{2n+1}(F_{2n+2}+1)}{F_{2n+3}} - \frac{F_{n+1}F_{n+3}}{F_{2n+3}} \begin{Bmatrix} 2n \\ n \end{Bmatrix}_F^{-1},$$

$$(ii)\ \sum_{k=0}^{n} F_{2(n-k)} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_F^{-2} = \frac{F_{2n+1}^2}{F_{2n+2}} - \frac{F_{n+1}}{L_{n+1}} \begin{Bmatrix} 2n \\ n \end{Bmatrix}_F^{2}.$$

In this paper, inspired by the results of [1] and earlier partial q-binomial sums formulae, we shall derive new kinds interesting *partial* sums formulae including the Gaussian q-binomial coefficients which are completely different from the sums formulae given in [1]. We summarize what we present in this paper below.

- *Sums of half* of the Gaussian q-binomial coefficients.
- *Partial sums of square* of the Gaussian q-binomial coefficients.
- *Partial sums of reciprocals* of the Gaussian q-binomial coefficients.
- *Partial sums of squared reciprocals* of the Gaussian q-binomial coefficients.

All above sums will be computed with certain weight functions. Further we notice that special cases of our results give us solutions for Advanced Problems H-764 and H-768 in [2,3]. All the identities and formulæ we will obtain hold for general q, and results about Fibonomial and Fibonacci numbers come out as corollaries for the special choice of q. One could derive many special corollaries by choosing special q values.

2. Upper Bound Cases

Now we present our results. Before this, we give an auxiliary lemma and then give our one of the main results.

Lemma 1. *For nonnegative integer n, any nonzero constant c and any function f,*

(i)

$$\sum_{j=0}^{n} c^{[2\nmid j]} f(j) = \sum_{j=0}^{n} \left(\frac{c+1-(c-1)(-1)^j}{2} \right) f(j),$$

(ii)

$$\sum_{j=0}^{n} c^{[2|j]} f(j) = \sum_{j=0}^{n} \left(\frac{c+1+(c-1)(-1)^j}{2} \right) f(j),$$

where $[\ ]$ stands for the Iverson notation.

Proof. For any integer j, since $[2|j] = \left(1+(-1)^j\right)/2$ and

$$\begin{aligned} c^{[2\nmid j]} f(j) &= [2|j]\, f(j) + c\,[2\nmid j]\, f(j) = ([2|j] + c\,(1-[2|j]))\, f(j) \\ &= (c-(c-1)\,[2|j])\, f(j) = \left(c+1-(c-1)\,(-1)^j\right) f(j)/2, \end{aligned}$$

the first claim (i) follows. The latter is similarly proven. □

Theorem 1. *(i) For even n,*

$$\sum_{j=0}^{n} \begin{bmatrix} 2n \\ j \end{bmatrix}_q \mathbf{i}^{j^2} (-1)^{j(n-1)} q^{\frac{j(j-2n+2)}{2}} (1-q^{2n-2j}) z^{[2\nmid j]}$$
$$= (-1)^{n+1}\, \mathbf{i}^{-n^2} q^{-\frac{1}{2}n^2+n-1} \frac{(1-q^n)(1-q^{n+1})}{1-q^{2n-1}} \begin{bmatrix} 2n \\ n \end{bmatrix}_q,$$

(ii) For odd n,

$$\sum_{j=0}^{n} \begin{bmatrix} 2n \\ j \end{bmatrix}_q \mathbf{i}^{j^2} (-1)^{j(n-1)} (1-q^{2n-2j}) z^{[2|j]}$$
$$= (-1)^{n+1}\, \mathbf{i}^{-n^2} q^{-\frac{1}{2}n^2+n-1} \frac{(1-q^n)(1-q^{n+1})}{1-q^{2n-1}} \begin{bmatrix} 2n \\ n \end{bmatrix}_q,$$

where $z = -\mathbf{i}q^{-1/2}(1+q)$.

Proof. We prove the claim (i). The latter is similar. Since n is even, say $n = 2k$, then we have to prove that

$$\sum_{j=0}^{2k} \begin{bmatrix} 4k \\ j \end{bmatrix}_q (-q)^{\frac{j(j-4k+2)}{2}} (1-q^{4k-2j}) \left(\frac{z+1-(z-1)(-1)^j}{2} \right)$$
$$= (-q)^{-2k^2+2k-1} \frac{(1-q^{2k})(1-q^{2k+1})}{(1-q^{4k-1})} \begin{bmatrix} 4k \\ 2k \end{bmatrix}_q.$$

Note that for any functions $F(j)$ and $G(j)$ of j, the following equality holds

$$\sum_{j=0}^{2n}\left[F(j)-G(j)\right]=\sum_{j=0}^{n}\left[F(2j)-G(2j)\right]+\sum_{j=0}^{n-1}\left[F(2j+1)-G(2j+1)\right].$$

By this fact, we rewrite the LHS of the claimed equality as

$$\begin{aligned}
&\sum_{j=0}^{2k}\begin{bmatrix}4k\\ j\end{bmatrix}_q(-q)^{\frac{j(j-4k+2)}{2}}(1-q^{4k-2j})\left(\frac{z+1-(z-1)(-1)^j}{2}\right)\\
&=\frac{1}{2}\sum_{j=0}^{2k}\begin{bmatrix}4k\\ j\end{bmatrix}_q(-q)^{\frac{j(j-4k+2)}{2}}(1-q^{4k-2j})\left(z+1-(z-1)(-1)^j\right)\\
&=\frac{(z+1)}{2}\sum_{j=0}^{k}\begin{bmatrix}4k\\ 2j\end{bmatrix}_q q^{2j(j-2k+1)}(1-q^{4k-4j})\\
&-\frac{(z-1)}{2}\sum_{j=0}^{k}\begin{bmatrix}4k\\ 2j\end{bmatrix}_q q^{2j(j-2k+1)}(1-q^{4k-4j})\\
&+\frac{1}{2}\sum_{j=0}^{k-1}\begin{bmatrix}4k\\ 2j+1\end{bmatrix}_q(-q)^{\frac{(2j+1)(2j-4k+3)}{2}}(1-q^{4k-4j-2})(z+1)\\
&+\frac{1}{2}\sum_{j=0}^{k-1}\begin{bmatrix}4k\\ 2j+1\end{bmatrix}_q(-q)^{\frac{(2j+1)(2j-4k+3)}{2}}(1-q^{4k-4j-2})(z-1)\\
&=\sum_{j=0}^{k}\begin{bmatrix}4k\\ 2j\end{bmatrix}_q q^{2j(j-2k+1)}(1-q^{4k-4j})\\
&+z\sum_{j=0}^{k-1}\begin{bmatrix}4k\\ 2j+1\end{bmatrix}_q(-q)^{\frac{(2j+1)(2j-4k+3)}{2}}(1-q^{4k-4j-2}),
\end{aligned}$$

which, by the identity

$$\begin{bmatrix}n+1\\ k+1\end{bmatrix}_q=\frac{1-q^{n-k+1}}{1-q^{k+1}}\begin{bmatrix}n+1\\ k\end{bmatrix}_q,$$

equals

$$\begin{aligned}
&\sum_{j=0}^{k}\begin{bmatrix}4k\\ 2j\end{bmatrix}_q q^{2j(j-2k+1)}(1-q^{4k-4j})\\
&+\sum_{j=0}^{k-1}\begin{bmatrix}4k\\ 2j\end{bmatrix}_q(-q)^{\frac{(2j+1)(2j-4k+3)}{2}}(1-q^{4k-4j-2})\frac{(1-q^{4k-2j})}{1-q^{2j+1}}z
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{k-1} \begin{bmatrix} 4k \\ 2j \end{bmatrix}_q q^{2j(j-2k+1)} (1 - q^{4k-4j}) \\
&+ \sum_{j=0}^{k-1} \begin{bmatrix} 4k \\ 2j \end{bmatrix}_q (-q)^{\frac{(2j+1)(2j-4k+3)}{2}} (1 - q^{4k-4j-2}) \frac{(1 - q^{4k-2j})}{1 - q^{2j+1}} z \\
&= \frac{-q^{1+2k-2k^2} (q;q)_{4k}}{(-q + q^{4k})(q;q)_{2k}(q;q)_{2k-2}} + \frac{q^{2k-2k^2}(1+q)(q;q)_{4k}}{(-q+q^{4k})(q;q)_{2k-1}^2}
\end{aligned}
$$

which, by the definition of the Gaussian q-binomial coefficients, equals

$$
\begin{aligned}
& q^{2k-2k^2} \frac{(1 - q^{2k-1})(1 - q^{2k})}{1 - q^{4k-1}} \begin{bmatrix} 4k \\ 2k \end{bmatrix}_q \\
& - q^{2k-2k^2-1}(1+q) \frac{(1 - q^{2k})(1 - q^{2k+1})}{1 - q^{4k-1}} \begin{bmatrix} 4k \\ 2k-1 \end{bmatrix}_q \\
&= q^{2k-2k^2} \frac{(1 - q^{2k-1})(1 - q^{2k})}{1 - q^{4k-1}} \begin{bmatrix} 4k \\ 2k \end{bmatrix}_q - q^{2k-2k^2-1}(1+q) \frac{(1 - q^{2k})^2}{1 - q^{4k-1}} \begin{bmatrix} 4k \\ 2k \end{bmatrix}_q \\
&= q^{2k-2k^2-1} \frac{1 - q^{2k}}{1 - q^{4k-1}} \begin{bmatrix} 4k \\ 2k \end{bmatrix}_q \left[q(1 - q^{2k-1}) - (1+q)(1 - q^{2k}) \right] \\
&= -q^{2k-2k^2-1} \frac{(1 - q^{2k})(1 - q^{2k+1})}{1 - q^{4k-1}} \begin{bmatrix} 4k \\ 2k \end{bmatrix}_q ,
\end{aligned}
$$

as claimed for even n such that $n = 2k$. The other claim is similarly proven. □

By taking $q = \left(p - \sqrt{p^2+4}\right) / \left(p + \sqrt{p^2+4}\right)$ in Theorem 1, we have the following result.

Corollary 1. *(i)For odd n,*

$$
\sum_{j=0}^{n} \begin{Bmatrix} 2n \\ j \end{Bmatrix}_U p^{[2|j]} U_{2n-2j} = \frac{U_n U_{n+1}}{U_{2n-1}} \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U .
$$

(ii)For even n,

$$
\sum_{j=0}^{n} \begin{Bmatrix} 2n \\ j \end{Bmatrix}_U p^{[2\nmid j]} U_{2n-2j} = \frac{U_n U_{n+1}}{U_{2n-1}} \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U .
$$

We notice that when $p = 1$, the results of Corollary 1 cover solutions for the advanced problem 764(i) [2].

Theorem 2. *For nonnegative integers n and m,*

$$\sum_{j=0}^{m} \begin{bmatrix} n \\ j \end{bmatrix}_q^2 q^{j(j-n+1)} \left(1-q^{n-2j}\right) = q^{m(m-n+1)} \left(1-q^{n-m}\right) \begin{bmatrix} n \\ m \end{bmatrix}_q \begin{bmatrix} n-1 \\ m \end{bmatrix}_q .$$

Proof. Denote the LHS of the claim by $F(n,m)$, that is,

$$F(n,m) = \sum_{j=0}^{m} \begin{bmatrix} n \\ j \end{bmatrix}_q^2 q^{j(j-n+1)} \left(1-q^{n-2j}\right).$$

For $m \geq n$, $F(n,m)$ is a whole sum and equals 0. To see this fact, consider

$$F(n,m) = \sum_{j\geq 0} \begin{bmatrix} n \\ j \end{bmatrix}_q^2 q^{j(j-n+1)} \left(1-q^{n-2j}\right),$$

which, by taking $n-j$ instead of j, gives us

$$F(n,m) = \sum_{j\geq 0} \begin{bmatrix} n \\ j \end{bmatrix}_q^2 q^{(j-n)(j-1)} \left(1-q^{2j-n}\right) = -F\left(n,m\right),$$

which gives us $F(n,m)=0$, as claimed.

Define

$$G(n,m) = q^{m(m-n+1)}(1-q^{n-m}) \begin{bmatrix} n \\ m \end{bmatrix}_q \begin{bmatrix} n-1 \\ m \end{bmatrix}_q .$$

Then we have

$$G(n,m) = F(n,m),$$

which follows from

$$\begin{aligned}
&G(n,j) - G(n,j-1) \\
&= q^{j(j-n+1)}(1-q^{n-j}) \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n-1 \\ j \end{bmatrix}_q - q^{(j-1)(j-n)}(1-q^{n-j+1}) \begin{bmatrix} n \\ j-1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q \\
&= q^{j(j-n+1)} \begin{bmatrix} n \\ j \end{bmatrix}_q \left((1-q^{n-j}) \frac{(q;q)_{n-1}}{(q;q)_j (q;q)_{n-j-1}} - \frac{q^{n-2j}(1-q^j)^2 (q;q)_{n-1}}{(1-q^{n-j})(q;q)_j (q;q)_{n-j-1}} \right) \\
&= q^{j(j-n+1)} \begin{bmatrix} n \\ j \end{bmatrix}_q \left(\frac{(1-q^{n-j})^2}{1-q^{cn}} \begin{bmatrix} n \\ j \end{bmatrix}_q - \frac{q^{n-2j}(1-q^j)^2}{1-q^n} \begin{bmatrix} n \\ j \end{bmatrix}_q \right) \\
&= q^{j(j-n+1)} \begin{bmatrix} n \\ j \end{bmatrix}_q^2 \left(\frac{(1-q^{n-j})^2 - q^{n-2j}(1-q^j)^2}{1-q^n} \right) \\
&= q^{j(j-n+1)} \begin{bmatrix} n \\ j \end{bmatrix}_q^2 \left(\frac{(1-q^{n-2j})(1-q^n)}{1-q^n} \right) \\
&= q^{j(j-n+1)}(1-q^{n-2j}) \begin{bmatrix} n \\ j \end{bmatrix}_q^2 ,
\end{aligned}$$

as claimed. □

As special cases of Theorem 2 with $q = \left(p-\sqrt{p^2+4}\right)/\left(p+\sqrt{p^2+4}\right)$, "$m \to n$, $n \to 2n$" and "$m = n$, $n \to 2n$", we have the following result, resp.

Corollary 2. *For nonnegative integer n,*

$$\sum_{j=0}^{n} \begin{Bmatrix} 2n \\ j \end{Bmatrix}_U^2 U_{2n-2j} = \frac{U_n}{V_n} \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U^2 \quad \text{and} \quad \sum_{j=0}^{2n} U_{2n-2j} \begin{Bmatrix} 2n \\ j \end{Bmatrix}_U^2 = 0.$$

When $p = 1$, the first result of Corollary 2 gives us a solution for the advanced problem 764(ii) [2].

Now we present a sum formula for the squared reciprocals of the Gaussian q-binomial coefficient without proof.

Theorem 3. *For nonnegative integers n and m,*

$$\sum_{k=0}^{m} \begin{bmatrix} n \\ k \end{bmatrix}_q^{-2} q^{k(n+1-k)} \left(1-q^{n-2k}\right) = \frac{\left(1-q^{n+1}\right)^2}{1-q^{n+2}} - q^{(m+1)(n-m)} \frac{\left(1-q^{m+1}\right)^2}{\left(1-q^{n+2}\right)} \begin{bmatrix} n \\ m \end{bmatrix}_q^{-2}.$$

As special cases of Theorem 3 with $q = \left(p-\sqrt{p^2+4}\right)/\left(p+\sqrt{p^2+4}\right)$, "$m \to n$, $n \to 2n$" and "$m = n$, $n \to 2n$", we have the following result, resp.

Corollary 3. *For nonnegative integer n,*

$$\sum_{j=0}^{n} U_{2n-2j} \begin{Bmatrix} 2n \\ j \end{Bmatrix}_U^{-2} = \frac{U_{2n+1}^2}{U_{2n+2}} - \frac{U_{n+1}}{V_{n+1}} \begin{Bmatrix} 2n \\ n \end{Bmatrix}_U^{-2} \quad \text{and} \quad \sum_{j=0}^{2n} U_{2n-2j} \begin{Bmatrix} 2n \\ j \end{Bmatrix}_U^{-2} = 0.$$

When $p = 1$, the first result of Corollary 3 gives a solution for the advanced problem 768(ii) [3].

3. Additional Sums Formulæ

In this Section, we will give new partial sums formulæ including the Gaussian q-binomial coefficients and their reciprocals.

Theorem 4. *For nonnegative integers n and m,*

(i)

$$\sum_{j=0}^{m} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{j(j+1-n)} \left(1-q^{n-2j}\right) = q^{m(m-n+1)} \left(1-q^{n-m}\right) \begin{bmatrix} n \\ m \end{bmatrix}_q.$$

(ii)

$$\sum_{j=0}^{m} \begin{bmatrix} n \\ j \end{bmatrix}_q^{-1} q^{j(n+1-j)} \left(1-q^{n-2j}\right) = \left(1-q^{n+1}\right) - q^{(m+1)(n-m)} \left(1-q^{m+1}\right) \begin{bmatrix} n \\ m \end{bmatrix}_q^{-1}.$$

Proof. As a showcase, we only prove the second formula. Denote

$$F(n,m)=\sum_{j=0}^{m}\begin{bmatrix}n\\j\end{bmatrix}_q^{-1}q^{j(n+1-j)}\left(1-q^{n-2j}\right).$$

For $m\geq n$, $F(n,m)$ is a whole sum and equals 0. To see this, consider by taking $n-j$ instead of j,

$$\begin{aligned}F(n,m)&=\sum_{j\geq 0}\begin{bmatrix}n\\j\end{bmatrix}_q^{-1}q^{j(n+1-j)}\left(1-q^{n-2j}\right)\\&=\sum_{j\geq 0}\begin{bmatrix}n\\j\end{bmatrix}_q^{-1}q^{(n-j)(j+1)}\left(1-q^{2j-n}\right)\\&=-F\left(n,m\right).\end{aligned}$$

So we get $F(n,m)=0$.

Define

$$G(n,m)=\left(1-q^{n+1}\right)-q^{(m+1)(n-m)}\left(1-q^{m+1}\right)\begin{bmatrix}n\\m\end{bmatrix}_q^{-1}.$$

Then we have

$$G(n,m)=F(n,m)$$

which follows from

$$\begin{aligned}&G(n,j)-G(n,j-1)\\&=\left(1-q^{n+1}\right)-q^{(j+1)(n-j)}\left(1-q^{j+1}\right)\begin{bmatrix}n\\j\end{bmatrix}_q^{-1}\\&-\left(\left(1-q^{n+1}\right)-q^{j(n-j+1)}\left(1-q^{j}\right)\begin{bmatrix}n\\j-1\end{bmatrix}_q^{-1}\right)\\&=-q^{(j+1)(n-j)}\left(1-q^{j+1}\right)\begin{bmatrix}n\\j\end{bmatrix}_q^{-1}+q^{j(n-j+1)}\left(1-q^{j}\right)\begin{bmatrix}n\\j-1\end{bmatrix}_q^{-1}\\&=-q^{(j+1)(n-j)}\left(1-q^{j+1}\right)\begin{bmatrix}n\\j\end{bmatrix}_q^{-1}+q^{j(n-j+1)}\left(1-q^{j}\right)\frac{(1-q^{n+1-j})}{(1-q^{j})}\begin{bmatrix}n\\j\end{bmatrix}_q^{-1}\\&=q^{j-j^2+jn}\left[q^{-2j+n}(-1+q^{j+1})+(1-q^{n+1-j})\right]\begin{bmatrix}n\\j\end{bmatrix}_q^{-1}\\&=q^{j(n+1-j)}(1-q^{n-2j})\begin{bmatrix}n\\j\end{bmatrix}_q^{-1}\end{aligned}$$

as claimed. □

References

[1] E. Kılıç and I. Akkus. On Fibonomial sums identities with special sign functions: analytically q-calculus approach. *accepted in Mathematica Slovaca.*

[2] H. Ohtsuka. Advanced Problem H-764. *The Fibonacci Quarterly.*, 52(4):375, 2014.

[3] H. Ohtsuka. Advanced Problem H-768. *The Fibonacci Quarterly.*, 53(1):89, 2015.

[4] Leonard Carlitz. The characteristic polynomial of a certain matrix of binomial coefficients. *Fibonacci Q.*, 3:81–89, 1965.

[5] Sergio Falcon. On the k-Lucas numbers. *Int. J. Contemp. Math. Sci.*, 6(21-24):1039–1050, 2011.

[6] Sergio Falcón and Ángel Plaza. On the Fibonacci k-numbers. *Chaos Solitons Fractals*, 32(5):1615–1624, 2007.

[7] H.W. Gould. The bracket function and Fontené-Ward generalized binomial coefficients with application to Fibonomial coefficients. *Fibonacci Q.*, 7:23–40, 1969.

[8] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete mathematics: a foundation for computer science. 2nd ed.* Amsterdam: Addison-Wesley Publishing Group, 2nd ed. edition, 1994.

[9] Verner E.jun. Hoggatt. Fibonacci numbers and generalized binomial coefficients. *Fibonacci Q.*, 5:383–400, 1967.

[10] Dov Jarden. Recurring sequences. *Math. Stud.*, 16:28–30, 1949.

[11] Emrah Kilic, Ilker Akkus, and Hideyuki Ohtsuka. Some generalized Fibonomial sums related with the Gaussian q-binomial sums. *Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér.*, 55(1):51–61, 2012.

[12] Emrah Kilic, Gabriela N. Stănică, and Pantelimon Stănică. Spectral properties of some combinatorial matrices. *Congr. Numerantium*, 201:223–235, 2010.

[13] Emrah Kilic and Pantelimon Stănică. Generating matrices of C-nomial coefficients and their spectra. In *Proceedings of the 14th international conference on Fibonacci numbers and their applications, Morelia, Mexico, July 5–9, 2010*, pages 139–154. México: Sociedad Matemática Mexicana, 2011.

[14] Emrah Kiliç. The generalized Fibonomial matrix. *Eur. J. Comb.*, 31(1):193–209, 2010.

[15] Emrah Kiliç. Evaluation of sums containing triple aerated generalized Fibonomial coefficients. *Math. Slovaca*, 67(2):355–370, 2017.

[16] Emrah Kiliç and Helmut Prodinger. Closed form evaluation of sums containing squares of Fibonomial coefficients. *Math. Slovaca*, 66(3):757–765, 2016.

[17] Emrah Kiliç and Helmut Prodinger. Evaluation of sums involving Gaussian q-binomial coefficients with rational weight functions. *Int. J. Number Theory*, 12(2):495–504, 2016.

[18] Emrah Kiliç, Helmut Prodinger, Ilker Akkus, and Hideyuki Ohtsuka. Formulas for fibonomial sums with generalized Fibonacci and Lucas coefficients. *Fibonacci Q.*, 49(4):320–329, 2011.

[19] Diego Marques and Pavel Trojovský. On some new sums of Fibonomial coefficients. *Fibonacci Q.*, 50(2):155–162, 2012.

[20] Diego Marques and Pavel Trojovský. On some new identities for the Fibonomial coefficients. *Math. Slovaca*, 64(4):809–818, 2014.

[21] Jaroslav Seibert and Pavel Trojovský. On some identities for the Fibonomial coefficients. *Math. Slovaca*, 55(1):9–19, 2005.

[22] R.F. Torretto and J.A. Fuchs. Generalized binomial coefficients. *Fibonacci Q.*, 2:296–302, 1964.

[23] Pavel Trojovský. On some identities for the Fibonomial coefficients via generating function. *Discrete Appl. Math.*, 155(15):2017–2024, 2007.

TOBB University of Economics and Technology, Mathematics Department, Sögütözü, 06560 Ankara, Turkey
E-mail address: ekilic@etu.edu.tr

Kirikkale University, Department of Mathematics, Faculty of Science and Arts, 71450 Kirikkale, Turkey
E-mail address: iakkus.tr@gmail.com