THE MATRIX OF SUPER PATALAN NUMBERS AND ITS FACTORIZATIONS

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Abstract. Matrices related to Patalan and super-Patalan numbers are factored according to the LU-decomposition. Results are obtained via inspired guessings and later proved using methods from Computer Algebra.

1. Introduction

Catalan numbers $C_n = \frac{1}{n+1} \binom{2n}{n}$ are very well known mathematical entities and even the subject of a whole book [3]. They can be generalized in at least two ways:

For integers $1 \leq q < p$, Richardson [2] defines $(q,p)$-Patalan numbers

$$b_n := -p^{2n+1} \binom{n-q/p}{n+1}.$$  

Here, the general definition of a binomial coefficient, $\binom{a}{k} := \frac{a(a-1)\ldots(a-k+1)}{k!}$ is employed. Now, for $q = 1, p = 2$ this leads to

$$b_n = -2^{2n+1} \binom{n-1/2}{n+1} = 2^{2n}(2n-1)(2n-3)\ldots1 \cdot \frac{(2n)!}{n!(n+1)!} = C_n,$$

which is a Catalan number.

In another direction, let

$$S(m, n) := \frac{(2m)!(2n)!}{m!n!(m+n)!},$$

a super Catalan number, then

$$S(m, 1) := \frac{(2m)!2}{m!(m+1)!} = 2C_n.$$  

Richardson [2] has generalized these as well via

$$Q(i, j) := (-1)^j p^{2(i+j)} \binom{i-q/p}{i+j},$$

again for integers $1 \leq q < p$. The special case arises, as before, for $q = 1, p = 2$:

$$Q(i, j) = (-1)^j 2^{2(i+j)} \binom{i-1/2}{i+j} = (-1)^j 2^{2(i+j)} \frac{(i-\frac{1}{2})(i-\frac{3}{2})\ldots(-j+\frac{1}{2})}{(i+j)!}$$

$$= (-1)^j 2^{i+j} \frac{(2i-1)(2i-3)\ldots(-2j+1)}{(i+j)!}$$

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\[
= 2^{i+j} \frac{2i-1)(2i-3)\ldots1(2j-1)(2j-3)\ldots1}{(i+j)!} \\
= \frac{(2i)!(2j)!}{(i+j)!i!j!} = S(i,j).
\]

The name \((p,q)\)-super Patalan numbers was chosen for the \(Q(i,j)\).

For a sequence \(a_n\), it is customary to arrange them in a matrix as follows:

\[
\begin{pmatrix}
  a_{0+r} & a_{1+r} & a_{2+r} & a_{3+r} & \cdots \\
  a_{1+r} & a_{2+r} & a_{3+r} & a_{4+r} & \cdots \\
  a_{2+r} & a_{3+r} & a_{4+r} & a_{5+r} & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Compare the general remarks in [1]. All our matrices are indexed starting at \((0,0)\) and have \(N\) rows resp. columns, where \(N\) might also be infinity, depending on the context. The nonnegative integer \(r\) is a \textit{shift parameter}.

Likewise, for a sequence \(a_{m,n}\), depending on two indices, one considers

\[
\begin{pmatrix}
  a_{0+r,0+s} & a_{0+r,1+s} & a_{0+r,2+s} & a_{0+r,3+s} & \cdots \\
  a_{1+r,0+s} & a_{1+r,1+s} & a_{1+r,2+s} & a_{1+r,3+s} & \cdots \\
  a_{2+r,0+s} & a_{2+r,1+s} & a_{2+r,2+s} & a_{2+r,3+s} & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

For any such matrix \(M\), we are interested in \textit{factorizations}, based on the \textit{LU}-decomposition: We use in a consistent way the notation \(M = LU\) and \(M^{-1} = AB\), and provide explicit expressions for \(L, L^{-1}, U, U^{-1}, A, A^{-1}, B, B^{-1}\).

This program will be executed for the matrix based on the sequence of \((q,p)\)-Patalan numbers as well as \((q,p)\)-super Patalan numbers, as well as for reciprocal \((q,p)\)-(super) Patalan numbers. Instead of working with the numbers \(p\) and \(q\), we find it easier to set \(x := \frac{q}{p}\), and our formulæ will work for general \(x\), provided that \(0 < x < 1\). Actually, they even work, provided \(x\) is not an integer.

We need the notion of falling factorials: \(x^2 := x(x-1)\ldots(x-n+1) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}\).

We only give proofs for the claimed results given in Sections 3 and 5. Others could be similarly done.

2. The Patalan matrix

In the next 4 sections, we list our findings. —

The matrix \(M\) has now entries

\[
M_{i,j} = -\frac{1}{p^{2(i+j+r)+1}} \binom{i+j+r-x}{i+j+r+1}.
\]

\[
L_{i,j} = \frac{(i+r-x)^{i-j}i!(2j+1+r)!}{p^{2i-2j}(i-j)!(i+j+1+r)!j!}
\]

\[
L^{-1}_{i,j} = \frac{(-1)^{i-j}i!(i+j+r)!(i+r-x)^{i-j}}{p^{2i-2j}(2i+r)!(i-j)!j!}
\]
The matrix \( B \) has now entries

\[
U_{i,j} = \frac{(j - x + r)^{i+j}(i + x)^{i+j+1}}{p^{2i+2j+1+2r}(i + j + 1 + r)!(j - i)!(2i + r)!} \\
U_{i,j}^{-1} = \frac{p^{2i+2j+1+2r}(i + j + 1 + r)!}{(i + r - x)^{i+j+r}(j + r)!!(j - i)!(2i + r)!}
\]

\[
A_{i,j} = \frac{(-1)^{i-j}p^{2i-2j}(i + 1 + r)!}{(x - 1 - 2j + r)^{i-j}(i + j + 1)!(j + 1 + r)!} \\
A_{i,j}^{-1} = \frac{p^{2i-2j}i!(i + 1 + r)!}{(i + j)!!(j + 1 + r)!(x + j - r)^{i-j}}
\]

\[
U_{i,j} = \frac{(-1)^{i+j+r}p^{2i+2j+1+2r}(j + 1 + r)!}{(x + 1 + r)^{j-i}(i + j)!!(j + 1)!}\] \\
\[U_{i,j}^{-1} = \frac{(-1)^{j-i}p^{2i+2j+1+2r}(i + 1)!}{(x - 1 - i + j - r)^{j-i}(j + r)!!(j - i)!(2i + r)!(x + j - r)^{j-i}}
\]

\[
A_{i,j} = \frac{(x - N + j - r)^{i-j}(i + 1)!}{p^{2i-2j}(N - 1 - i)!!(i + j + 1)!!(i - j)!}
\]

\[
A_{i,j}^{-1} = \frac{(-1)^{i-j}(x - N + j - r)^{i-j}(i + 1)!}{p^{2i-2j}(2i + 1)!!(N - 1 - i)!(i + j + 1)!!(i - j)!(2i + r)!(x + j - r)^{i-j}}
\]

\[
B_{i,j} = \frac{(-1)^{i-j}(x + 1)!}{p^{2i+2j+1+2r}(N - 1 - j)!(j + 2 + r)!!(j - i)!(2i + 1 + r)!}
\]

\[
B_{i,j}^{-1} = \frac{p^{2i+2j+1+2r}(x + 1)!(i + j + 1)!!(i + 1)!}{(N + i - 1 + r - x)^{i+j+2+r}(N + j + r)!(j - i)!}
\]

3. The reciprocal Patalan matrix

The matrix \( M \) has now entries

\[
M_{i,j} = (-1)^{i+j+r}p^{2(i+j+r)}\left(\frac{r}{i + r + x}\right)^{i+j+r}.
\]
\[ L_{i,j} = \frac{p^{2i-2j}(x - i + r)_{i-j}!}{i!(i + r + j)!j!(i + j + r + s)!} \]
\[ L_{i,j}^{-1} = \frac{p^{2i-2j}(x - j - 1 - r)_{i-j}!}{(2i - 1 + r + s)!i!(i + j - 1 + r + s)!j!} \]
\[ U_{i,j} = \frac{(-1)^{j+1+r+s}p^{2i+2j+2r+2s}(x - j - 1)_{j-i}!(1 + i + j + r + s)!}{(j - i)!(2i - 1 + r + s)!i!(i + j + r + s)!} \]
\[ U_{i,j}^{-1} = \frac{(-1)^{s}(2j + r + s)!}{p^{2i+2j+2r+2s}(x - j)!i!(i + j - 1 + r + s)!} \]
\[ A_{i,j} = \frac{p^{2i-2j}(x - i - 1 + s)!}{(N + i - 1 + r + s)!i!(i + j - 1 + r + s)!} \]
\[ A_{i,j}^{-1} = \frac{(-1)^{N+i+1+r+s}(N - 1 - i)!}{p^{2i+2j+2r+2s}(x - j)!i!(i + j - 1 + r + s)!} \]

5. The reciprocal super Patalan matrix

The matrix \( M_{i,j} \) has now entries

\[ M_{i,j} = (-1)^{j+s}p^{2(i+r+j+s)} \left( \begin{array}{c} i + r - x \\ i + r + j + s \end{array} \right)^{-1} \]

\[ L_{i,j} = \frac{i!(i + r + s)!}{p^{2i-2j}(x + i + r)_{i-j}!j!(j + r + s)!} \]
\[ L_{i,j}^{-1} = \frac{i!(i + r + s)!}{p^{2i+2j+2r+2s}(x - 1)_{i-j}!(1 + i + j + r + s)!} \]
\[ U_{i,j} = \frac{(-1)^{i+j+1+r+s}(j + r + s)!}{p^{2i+2j+2r+2s}(x - j - 1)_{j-i}!(1 + i + j + r + s)!} \]
\[ U_{i,j}^{-1} = \frac{(-1)^{i+j+1+r+s}}{p^{2i+2j+2r+2s}(x - 1)_{i-j}!(1 + i + j + r + s)!} \]
\[ A_{i,j} = \frac{p^{2i-2j}(2j + 1 + r + s)!}{(N - 1 - j)!(1 + i + j + r + s)!j!(i + j - 1 + r + s)!} \]
\[ A_{i,j}^{-1} = \frac{p^{2i-2j}(i + j + r + s)!}{(N - 1 - j)!(1 + i + j + r + s)!j!(i + j - 1 + r + s)!} \]

\[ B_{i,j} = \frac{(-1)^{i+j+s}p^{2i+2j+2r+2s}(x + j + r)!}{(N - 1 - j)!(1 + i + j + r + s)!j!(i + j - 1 + r + s)!} \]
\[ B_{i,j}^{-1} = \frac{(-1)^{i+j+s}}{p^{2i+2j+2r+2s}(x - 1)!i!(i + j + r + s)!} \]
6. Proofs for the results of Section 3

For $L$ and $L^{-1}$, we should prove the following equation

$$\sum_{j \leq d \leq i} L_{id} L^{-1}_{dj} = \delta_{i,j},$$

where $\delta_{k,j}$ is the Kronecker delta. So we have

$$\sum_{j \leq d \leq i} L_{id} L^{-1}_{dj} = (-1)^i p^{2i-2j} \frac{(i+1+r)!}{(j+1+r)!j!} \times \sum_{j \leq d \leq i} (-1)^d \left( \frac{x - 1 - 2d - r}{i - d} \right)^{-1} \left( \frac{x - d - j - r}{d - j} \right)^{-1} \frac{1}{((d-j)!)^2 ((i-d)!)^2},$$

which equals 0 when $i \neq j$ by Zeilberger’s algorithm. The case $L_{ii} L^{-1}_{ii} = 1$ can be directly seen.

For $U$ and $U^{-1}$, we have

$$\sum_{i \leq d \leq j} U_{id} U^{-1}_{dj} = (-1)^i p^{2i-2j} \frac{x^{2j+1+r}}{(x+1)^j (x-i+1)^{2j+2+r}} \times \sum_{i \leq d \leq j} (-1)^d \left( \frac{x - i + 1}{d + 2 + r} \right)^{-1} \left( \frac{x - j - r}{d} \right) \frac{d!}{(d+2+r)! (d-i)! (j-d)!},$$

The Zeilberger algorithm computes that the previous sum is equal to 0 when $i \neq j$. If $i = j$, we get

$$U_{ii} U^{-1}_{ii} = \frac{(x+1)^{2i+1+r}}{(x+1)^{i+1} (x-i+1)^{2i+2+r}} = 1,$$

which completes the proof.

For the $LU$-decomposition, we have to prove that

$$\sum_{0 \leq d \leq \min\{i,j\}} L_{id} U_{dj} = M_{ij}.$$ 

Consider

$$\sum_{0 \leq d \leq \min\{i,j\}} L_{id} U_{dj} = p^{2i+2j+2r+1} \sum_{0 \leq d \leq \min\{i,j\}} \left( \frac{x - 1 - 2d - r}{i - d} \right)^{-1} \left( \frac{x - d + 1}{j + 2 + r} \right)^{-1} \left( \frac{x - d - r}{d} \right)^{-1} \times \frac{(-1)^{i+j+r} (i+1+r)! (x+1) j!}{(j+r+2) ((i-d)!)^2 (d+1+r)! (j-d)!}. $$

Without loss of generality, we choose $i \leq j$. Denote the RHS of the sum in the equation just above by $\text{SUM}_i$. The Mathematica version of the Zeilberger algorithm produces the recursion

$$\text{SUM}_i = \frac{j + i + r + 1}{i + j + r - x} \text{SUM}_{i-1}. $$
Since \( \text{SUM}_0 = \frac{(-1)^{i+j} (j+r+1)! (x+1)}{x+1} \), we obtain

\[
\text{SUM}_i = \frac{(j + i + r + 1)^2 (-1)^{i+j} (j + r + 1)! (x + 1)}{(i + j + r - x)^2 (x + 1)^{i+j+r+1}} \frac{(i + j + r + 1)!}{(i + j + r - x)^{i+j+r+1}} \]

\[
= -\frac{(i + j + r + 1)!}{(i + j + r - x)^{i+j+r+1}} \frac{(i + j + r + 1)!}{(i + j + r + 1)^{i+j+r+1}} \]

So we get

\[
\sum_{0 \leq d \leq \min\{i,j\}} L_{id} U_{dj} = M_{kj},
\]

as claimed.

For \( A \) and \( A^{-1} \), we have

\[
\sum_{j \leq d \leq i} A_{id} A_{dj}^{-1} = p^{2j-2k} \frac{(-1)^j (N - 1 - j)!}{(N - 1 - i)!} \sum_{j \leq d \leq i} (-1)^d \frac{(2d + 2 + r)}{(i + d + 2 + r)!} \left( \frac{x - N - d - r}{i - d} \right) \left( \frac{x - N - j - r}{d - j} \right),
\]

which equals 0 provided that \( i \neq j \). If \( i = j \), it is obvious that \( A_{ii} A_{ii}^{-1} = 1 \). Thus

\[
\sum_{i \leq d \leq j} A_{id} A_{dj}^{-1} = \delta_{i,j},
\]

as claimed.

For \( B \) and \( B^{-1} \), by using the Zeilberger algorithm, similarly we obtain

\[
\sum_{i \leq d \leq j} B_{id} B_{dj}^{-1} = \delta_{i,j}.
\]

For the LU-decomposition of \( M^{-1} \), we should prove that \( M^{-1} = AB \) which is same as \( M = B^{-1} A^{-1} \). So it is sufficient to show that

\[
\sum_{\max\{i,j\} \leq d \leq n-1} B_{id}^{-1} A_{dj}^{-1} = M_{ij}.
\]

After some rearrangements, we have

\[
\sum_{j \leq d \leq n-1} B_{id}^{-1} A_{dj}^{-1} = p^{2i+2j+2r+1} \sum_{j \leq d \leq n-1} (-1)^{d-j} \left( \frac{x - N - j - r}{d - j} \right) \left( \frac{N - 1 + i + r - x}{i + d + 2 + r} \right)^{-1} \left( \frac{x + 1}{N + 1 + d + r} \right)\left( \frac{(N - 1 - i)! (d + j + 1 + r)! (N - 1 - j)!}{(d - i)! (N - 1 - d)! (i + d + 2 + r)} \right).
\]
Here we replace \((N - 1)\) with \(N\) and denote the RHS of the sum by \(\text{SUM}_N\). The Zeilberger algorithm produces the recursion
\[
\text{SUM}_N = \text{SUM}_{N-1}.
\]
So
\[
\text{SUM}_N = \text{SUM}_N = \frac{(x + 1)(i + j + 1 + r)!}{(j + i + r - x)^{i+j+2+r}}
= \frac{(x + 1)(i + j + 1 + r)!}{(j + i + r - x)^{i+j+1+r}} (-x - 1)^{-1}
\]
which completes the proof.

7. Proofs for the results of Section 5

For \(L\) and \(L^{-1}\), we have
\[
\sum_{j \leq d \leq i} L_{id}L_{d,j}^{-1} = p^{2i-2j} \frac{i!}{j!} \sum_{j \leq d \leq i} \left( \frac{i + r - x}{i - d} \right) \left( \frac{x - j - 1 - r}{d - j} \right)
\[
\times \frac{(2d + r + s)(j + d + r + s - 1)!}{(i + d + r + s)!}
\]
By the Zeilberger algorithm, the sum of the RHS of the equation just above is equal to 0 when \(i \neq j\). The case \(i = j\) can be easily computed. So
\[
\sum_{j \leq d \leq i} L_{id}L_{d,j}^{-1} = \delta_{i,j},
\]
as desired.

For \(U\) and \(U^{-1}\),
\[
\sum_{i \leq d \leq j} U_{id}U_{d,j}^{-1} = \frac{(-1)^{r+1} p^{2i-2j} x^{i+1+r} (i + r + s - 1)! (2j + r + s)!}{(2k - 1 + r + s)! (j - 1 + r + s)!}
\]
\[
\times \sum_{i \leq d \leq j} (-1)^d \left( \frac{-x - 1}{d - 1 + s} \right) \left( \frac{-x + j + r}{d + j + r + s} \right)
\]
\[
\times \frac{(d - 1 - s)!}{(d + j + r + s)(d - i)! (j - d)! (i + d + r + s)!},
\]
which equals 0 provided that \((j - i)(j + i + r + s) \neq 0\). Since \(r\) and \(s\) are nonnegative integer parameters, only the case \(j = i\) should be examined. Consider
\[
U_{ii}U_{ii}^{-1} = \frac{(-1)^{r+1} x^{i+1+r} (-x - 1)^{i+s-1}}{(-x + i + r)^{2i+r+s}} = 1,
\]
which completes the proof.

Similarly, for \(LU\)-decomposition, we have to prove that
\[
\sum_{0 \leq d \leq \min\{i,j\}} L_{id}U_{d,j} = M_{ij}.
\]
So without loss of generality we may choose \( i \leq j \). Then we obtain

\[
\sum_{0 \leq d \leq i} L_{id} U_{dj} = p^{2(i+j+r+s)} (-1)^{j+1+r+s} \sum_{0 \leq d \leq i} \binom{-x + i + r}{i - d} \binom{x}{d + 1 + r} \times \frac{(-x - 1)!}{(j + s - 1)!} \frac{(2d + r + s) (d + r + s - 1)! (d + 1 + r)!}{d! (i + d + r + s) (j - d)! (d + j + r + s)!}.
\]

Denote the above sum on the RHS in the equation above by \( \text{SUM}_i \), the Zeilberger algorithm produces the recurrence relation for \( \text{SUM}_i \):

\[
\text{SUM}_i = \frac{i + r - x}{i + j + r + s} \text{SUM}_{i-1},
\]

with the initial \( \text{SUM}_0 = \binom{x}{1+r} \binom{-x-1}{j+s-1} \). If we solve the recurrence, we obtain

\[
\text{SUM}_i = \left( \frac{i + r - x}{i + j + r + s} \right)^i \text{SUM}_0 = (-1)^{1+r} \binom{i + r - x}{i + r + j + s},
\]

which completed the proof.

For \( A \) and \( A^{-1} \), consider

\[
\sum_{j \leq d \leq i} A_{id} A_{dj}^{-1} = p^{2j-2k} \frac{(N + i + r + s - 1)! (N - 1 - j)!}{(N + j + r + s - 1)! (N - 1 - i)!} \times \sum_{j \leq d \leq i} \binom{x + d - 1 + s}{d - j}^{-1} \binom{-x - s - d}{i - d}^{-1} \frac{1}{((d-j))!^2 ((i-d))!^2}.
\]

When \( j \neq i \), the Zeilberger algorithm evaluates that the sum on the RHS in the equation above is equal to 0. The case \( j = i \) can be easily computed.

Similarly, we have

\[
\sum_{i \leq d \leq j} B_{id} B_{dj}^{-1} = p^{2j-2k} \binom{-x}{i+r} \binom{-x}{j+r} \sum_{i \leq d \leq j} \frac{(-1)^d}{(j-d)! (d-k)!} = \delta_{i,j}.
\]

Finally, by using the same argument in previous section, we should show that

\[
\sum_{\max\{i,j\} \leq d \leq N-1} B_{id} A_{dj}^{-1} = M_{ij}.
\]

Consider \( j \geq i \), then we have

\[
\sum_{j \leq d \leq N-1} B_{id}^{-1} A_{dj}^{-1} = p^{2(i+j+r+s)} (x - 1)^{i+r} \binom{-x}{k+r+s} \sum_{j \leq d \leq N-1} (-1)^{N+1} \binom{-x}{d+s} \binom{x + d - 1 + s}{d - j}^{-1} \times \frac{(d + s)! (N - 1 - i)! (N - 1 - j)! (N - 1 + d + r + s)!}{((d-j))!^2 (d-i)! (N-1+i+r+s)! (N-1-d)! (N-1+j+r+s)!}.
\]

The case \( j = i \) can be easily computed.
By replacing \((N - 1)\) with \(N\) on the RHS in the equation just above, we obtain the sum

\[
\sum_{j \leq d \leq N} (-1)^N \cdot (-x)^d \cdot \binom{x + d - 1 + s}{d - j}^{-1} 
\times \frac{(d + s)! (N - i)! (N - j)! (N + d + r + s)!}{((d - j)!)^2 (d - i)! (N + i + r + s)! (N - d)! (N + j + r + s)!}.
\]

Denote this sum by \(\text{SUM}_N\). By using the Zeilberger algorithm, we get

\[
\text{SUM}_N = \text{SUM}_{N-1} = \text{SUM}_j = \frac{(-1)^j (-x)^j}{(j + i + r + s)!}.
\]

Summarizing,

\[
\sum_{j \leq d \leq N-1} B_{id}^{-1} A_{dj}^{-1} = p^{2(i+j+r+s)} (x - 1)^{i+r} (-1)^{i+r+s} \frac{(-1)^j (-x)^j}{(j + i + r + s)!} 
\times (-1)^{j+s} p^{2(i+r+j+s)} \binom{i + r - x}{i + r + j + s},
\]

as claimed.

REFERENCES


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