THE MATRIX OF SUPER PATALAN NUMBERS AND ITS FACTORIZATIONS

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Abstract. Matrices related to Patalan and super-Patalan numbers are factored according to the LU-decomposition. Results are obtained via inspired guessings and later proved using methods from Computer Algebra.

1. Introduction

Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ are very well known mathematical entities and even the subject of a whole book [3]. They can be generalized in at least two ways:

For integers $1 \le q < p$, Richardson [2] defines (q,p)-Patalan numbers

$$b_n := -p^{2n+1}\binom{n-q/p}{n+1}.$$

Here, the general definition of a binomial coefficient, $\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}$ is employed. Now, for $q=1$, $p=2$ this leads to

$$b_n = -2^{2n+1}\binom{n-1/2}{n+1} = 2^n\frac{(2n-1)(2n-3)\dots 3\cdot 1}{(n+1)!} = \frac{(2n)!}{n!(n+1)!} = C_n,$$

which is a Catalan number.

In an other direction, let

$$S(m,n) := \frac{(2m)!(2n)!}{m!n!(m+n)!},$$

a *super Catalan number*, then

$$S(m,1) := \frac{(2m)!2}{m!(m+1)!} = 2C_n.$$

Richardson [2] has generalized these as well via

$$Q(i,j) := (-1)^j p^{2(i+j)}\binom{i-q/p}{i+j},$$

again for integers $1 \le q < p$. The special case arises, as before, for $q=1$, $p=2$:

$$\begin{aligned}Q(i,j) &= (-1)^j 2^{2(i+j)}\binom{i-1/2}{i+j} = (-1)^j 2^{2(i+j)}\frac{(i-\frac12)(i-\frac32)\dots(-j+\frac12)}{(i+j)!}\\ &= (-1)^j 2^{i+j}\frac{(2i-1)(2i-3)\dots(-2j+1)}{(i+j)!}\end{aligned}$$

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$$= 2^{i+j}\frac{(2i-1)(2i-3)\dots 1(2j-1)(2j-3)\dots 1}{(i+j)!}$$
$$= \frac{(2i)!(2j)!}{(i+j)!i!j!} = S(i,j).$$

The name (p,q)-super Patalan numbers was chosen for the $Q(i,j)$.

For a sequence a_n, it is customary to arrange them in a matrix as follows:

$$\begin{pmatrix} a_{0+r} & a_{1+r} & a_{2+r} & a_{3+r} & \dots \\ a_{1+r} & a_{2+r} & a_{3+r} & a_{4+r} & \dots \\ a_{2+r} & a_{3+r} & a_{4+r} & a_{5+r} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Compare the general remarks in [1]. All our matrices are indexed starting at $(0,0)$ and have N rows resp. columns, where N might also be infinity, depending on the context. The nonnegative integer r is a *shift parameter*.

Likewise, for a sequence $a_{m,n}$, depending on two indices, one considers

$$\begin{pmatrix} a_{0+r,0+s} & a_{0+r,1+s} & a_{0+r,2+s} & a_{0+r,3+s} & \dots \\ a_{1+r,0+s} & a_{1+r,1+s} & a_{1+r,2+s} & a_{1+r,3+s} & \dots \\ a_{2+r,0+s} & a_{2+r,1+s} & a_{2+r,2+s} & a_{2+r,3+s} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

For any such matrix M, we are interested in *factorizations*, based on the *LU*-decomposition: We use in a consistent way the notation $M = LU$ and $M^{-1} = AB$, and provide explicit expressions for L, L^{-1}, U, U^{-1}, A, A^{-1}, B, B^{-1}.

This program will be executed for the matrix based on the sequence of (q,p)-Patalan numbers as well as (q,p)-super Patalan numbers, as well as for reciprocal (q,p)-(super) Patalan numbers. Instead of working with the numbers p and q, we find it easier to set $x := \frac{q}{p}$, and our formulæ will work for general x, provided that $0 < x < 1$. Actually, they even work, provided x is not an integer.

We need the notion of falling factorials: $x^{\underline{n}} := x(x-1)\dots(x-n+1) = \frac{\Gamma((x+1)}{\Gamma(x-n+1)}$.

We only give proofs for the claimed results given in Sections 3 and 5. Others could be similarly done.

2. The Patalan matrix

In the next 4 sections, we list our findings. —

The matrix M has now entries

$$M_{i,j} = -\frac{1}{p^{2(i+j+r)+1}}\binom{i+j+r-x}{i+j+r+1}.$$

$$L_{i,j} = \frac{(i+r-x)^{\underline{i-j}}i!(2j+1+r)!}{p^{2i-2j}(i-j)!(i+j+1+r)!j!}$$
$$L^{-1}_{i,j} = \frac{(-1)^{i-j}i!(i+j+r)!(i+r-x)^{\underline{i-j}}}{p^{2i-2j}(2i+r)!(i-j)!j!}$$

$$U_{i,j} = \frac{(j-x+r)^{\underline{j+r}}(i+x)^{\underline{i+1}}j!(i+r)!}{p^{2i+2j+1+2r}(i+j+1+r)!(j-i)!(2i+r)!}$$

$$U^{-1}_{i,j} = \frac{p^{2i+2j+1+2r}(-1)^{i-j}(2j+1+r)!(i+j+r)!}{(j+x)^{\underline{j+1}}(i+r-x)^{i+r}(j+r)!(j-i)!i!}$$

$$A_{i,j} = \frac{p^{2i-2j}(N+i+r)!(N-1-j)!}{(N-1-i)!(i-j)!(N+j+r)!(x-r-2j-2)^{\underline{i-j}}}$$

$$A^{-1}_{i,j} = \frac{(-1)^{i+j}p^{2i-2j}(N+i+r)!(N-1-j)!}{(N-1-i)!(i-j)!(N+j+r)!(x-1-i-j-r)^{\underline{i-j}}}$$

$$B_{i,j} = \frac{(-1)^{i+j+N+1+r}p^{2i+2j+1+2r}(N-j+r)!(x-i-j-2-r)^{\underline{N-1-j}}}{(N-i-1+x)^{\underline{N+i+r}}(N-1-j)!(j-i)!}$$

$$B^{-1}_{i,j} = \frac{(-1)^{i+j+1+r}(N-1-i)!(-x)^{\underline{N-j}}(x-1)^{\underline{i+j+r}}}{p^{2i+2j+1+2r}(j-i)!(N+i+r)!(x-2j-2-r)^{\underline{N-1-j}}}$$

3. The reciprocal Patalan matrix

The matrix M has now entries

$$M_{i,j} = -p^{2(i+j+r)+1}\binom{i+j+r-x}{i+j+r+1}^{-1}.$$

$$L_{i,j} = \frac{(-1)^{i-j}p^{2i-2j}(i+1+r)!i!}{(x-1-2j-r)^{\underline{i-j}}(i-j)!j!(j+1+r)!}$$

$$L^{-1}_{i,j} = \frac{p^{2i-2j}i!(i+1+r)!}{(i-j)!j!(j+1+r)!(x-i-j-r)^{\underline{i-j}}}$$

$$U_{i,j} = \frac{(-1)^{i+j+r}p^{2i+2j+1+2r}(j+1+r)!(x+1)j!}{(j-i)!(x-i+1)^{\underline{j+2+r}}(x-i-r)^{\underline{i}}}$$

$$U^{-1}_{i,j} = \frac{(-1)^{r}(x-j-r)^{\underline{i}}x^{\underline{2j+1+r}}}{p^{2i+2j+1+2r}(x+1)^{\underline{j}}i!(i+1+r)!(j-i)!}$$

$$A_{i,j} = \frac{(x-N-j-r)^{\underline{i-j}}(2j+2+r)!(N-1-j)!}{p^{2i-2j}(N-1-i)!(i+j+2+r)!(i-j)!}$$

$$A^{-1}_{i,j} = \frac{(-1)^{i-j}(x-N-j-r)^{\underline{i-j}}(i+j+1+r)!(N-1-j)!}{p^{2i-2j}(2i+1+r)!(N-1-i)!(i-j)!}$$

$$B_{i,j} = \frac{(-1)^{r}(x-N+i+2)^{\underline{i+j+2+r}}(N+1+i+r)!}{p^{2i+2j+1+2r}(x+1)(N-1-j)!(i+j+2+r)!(j-i)!(2i+1+r)!}$$

$$B^{-1}_{i,j} = \frac{p^{2i+2j+1+2r}(x+1)(2j+2+r)!(i+j+1+r)!(N-1-i)!}{(N+i-1+r-x)^{\underline{i+j+2+r}}(N+1+j+r)!(j-i)!}$$

4. The super Patalan matrix

The matrix M has now entries

$$M_{i,j} = (-1)^{j+s}p^{2(i+r+j+s)}\binom{i+r-x}{i+r+j+s}.$$

$$L_{i,j} = \frac{p^{2i-2j}(-x+i+r)^{\underline{i-j}}\, i!(2j+r+s)!}{(i-j)!(i+j+r+s)!j!}$$

$$L^{-1}_{i,j} = \frac{p^{2i-2j}(x-j-1-r)^{\underline{i-j}}(i+j-1+r+s)!i!}{(2i-1+r+s)!(i-j)!j!}$$

$$U_{i,j} = \frac{(-1)^{j+1+r+s}p^{2i+2j+2r+2s}(-x-1)^{\underline{j-1+s}}x^{\underline{i+1+r}}j!(i+r+s-1)!}{(j-i)!(2i-1+r+s)!(i+j+r+s)!}$$

$$U^{-1}_{i,j} = \frac{(-1)^{s}(2j+r+s)!(i+j-1+r+s)!}{p^{2i+2j+2r+2s}(-x+j+r)^{\underline{i+j+r+s}}i!(j-i)!(j-1+r+s)!}$$

$$A_{i,j} = \frac{(N+i-1+r+s)!(N-1-j)!}{p^{2i-2j}(-x-s-j)^{\underline{i-j}}(N-1-i)!(i-j)!(N+j-1+r+s)!}$$

$$A^{-1}_{i,j} = \frac{(N+i-1+r+s)!(N-1-j)!}{p^{2i-2j}(x+i-1+s)^{\underline{i-j}}(N-1-i)!(i-j)!(N+j-1+r+s)!}$$

$$B_{i,j} = \frac{(-1)^{N+i+1+r+s}(N+j-1+r+s)!}{p^{2i+2j+2r+2s}(x-1)^{\underline{j+r}}(-x)^{\underline{i+s}}(N-1-j)!(j-i)!}$$

$$B^{-1}_{i,j} = \frac{(-1)^{N+i+1+r+s}p^{2i+2j+2r+2s}(-x)^{\underline{j+s}}(x-1)^{\underline{i+r}}(N-1-i)!}{(j-i)!(N+i-1+r+s)!}$$

5. The reciprocal super Patalan matrix

The matrix M has now entries

$$M_{i,j} = (-1)^{j+s}p^{2(i+r+j+s)}\binom{i+r-x}{i+r+j+s}^{-1}.$$

$$L_{i,j} = \frac{i!(i+r+s)!}{p^{2i-2j}(-x+i+r)^{\underline{i-j}}(i-j)!j!(j+r+s)!}$$

$$L^{-1}_{i,j} = \frac{i!(i+r+s)!}{p^{2i-2j}(x-j-1-r)^{\underline{i-j}}(i-j)!j!(j+r+s)!}$$

$$U_{i,j} = \frac{(-1)^{i+j+1+r+s}(j+r+s)!j!}{p^{2i+2j+2r+2s}(-x-1)^{\underline{j-1+s}}x^{\underline{i+1+r}}(j-i)!}$$

$$U^{-1}_{i,j} = \frac{(-1)^{r+s+1}p^{2i+2j+2r+2s}x^{\underline{j+1}}(x-j-1)^{\underline{r}}(-x-1)^{\underline{i-1+s}}}{i!(j-i)!(i+r+s)!}$$

$$A_{i,j} = \frac{p^{2i-2j}(2j+1+r+s)!(N-1-j)!(-x-j-s)^{\underline{i-j}}}{(i+j+1+r+s)!(i-j)!(N-1-i)!}$$

$$A^{-1}_{i,j} = \frac{p^{2i-2j}(i+j+r+s)!(N-1-j)!(x-1+i+s)^{\underline{i-j}}}{(N-1-i)!(2i+r+s)!(i-j)!}$$

$$B_{i,j} = \frac{(-1)^{j+s}p^{2i+2j+2r+2s}(-x+j+r)^{\underline{i+j+r+s}}(N+i+r+s)!}{(N-1-j)!(j-i)!(i+j+1+r+s)!(2i+r+s)!}$$

$$B^{-1}_{i,j} = \frac{(-1)^{j+s}(N-1-i)!(i+j+r+s)!(2j+1+r+s)!}{p^{2i+2j+2r+2s}(j-i)!(N+j+r+s)!(-x+i+r)^{\underline{i+j+r+s}}}$$

6. Proofs for the results of Section 3

For L and L^{-1}, we should prove the following equation

$$\sum_{j\le d\le i} L_{id}L^{-1}_{dj} = \delta_{i,j},$$

where $\delta_{k,j}$ is the Kronecker delta. So we have

$$\sum_{j\le d\le i} L_{id}L^{-1}_{dj} = (-1)^i p^{2i-2j}\frac{(i+1+r)!i!}{(j+1+r)!j!}$$
$$\times \sum_{j\le d\le i}(-1)^d\binom{x-1-2d-r}{i-d}^{-1}\binom{x-d-j-r}{d-j}^{-1}\frac{1}{((d-j)!)^2((i-d)!)^2},$$

which equals 0 when $i\ne j$ by Zeilberger's algorithm. The case $L_{ii}L^{-1}_{ii}=1$ can be directly seen.

For U and U^{-1}, we have

$$\sum_{i\le d\le j} U_{id}U^{-1}_{dj} = \frac{(-1)^i p^{2i-2j}x^{\underline{2j+1+r}}(x+1)}{(x+1)^{\underline{j}}(x-k-r)^{\underline{i}}}\sum_{i\le d\le j}(-1)^d\binom{x-i+1}{d+2+r}^{-1}\binom{x-j-r}{d}$$
$$\times\frac{d!}{(d+2+r)!\,(d-i)!\,(j-d)!}.$$

The Zeilberger algorithm computes that the previous sum is equal to 0 when $i\ne j$. If $i=j$, we get

$$U_{ii}U^{-1}_{ii} = \frac{(x+1)\,x^{\underline{2i+1+r}}}{(x+1)^{\underline{i}}(x-i+1)^{\underline{i+2+r}}} = 1,$$

which completes the proof.

For the LU-decomposition, we have to prove that

$$\sum_{0\le d\le \min\{i,j\}} L_{id}U_{dj} = M_{ij}.$$

Consider

$$\sum_{0\le d\le\min\{i,j\}} L_{id}U_{dj}$$
$$= p^{2i+2j+2r+1}\sum_{0\le d\le\min\{i,j\}}\binom{x-1-2d-r}{i-d}^{-1}\binom{x-d+1}{j+2+r}^{-1}\binom{x-d-r}{d}^{-1}$$
$$\times\frac{(-1)^{i+j+r}(i+1+r)!i!\,(x+1)\,j!}{(j+r+2)\,((i-d)!)^2\,(d!)^2\,(d+1+r)!\,(j-d)!}.$$

Without loss of generality, we choose $i\le j$. Denote the RHS of the sum in the equation just above by $\texttt{SUM}_i$. The Mathematica version of the Zeilberger algorithm produces the recursion

$$\texttt{SUM}_i = \frac{j+i+r+1}{i+j+r-x}\texttt{SUM}_{i-1}.$$

Since $\mathtt{SUM}_0 = \frac{(-1)^{j+r}(j+r+1)!(x+1)}{(x+1)^{\underline{j+2+r}}}$, we obtain

$$\begin{aligned}\mathtt{SUM}_i &= \frac{(j+i+r+1)^{\underline{i}}}{(i+j+r-x)^{\underline{i}}} \frac{(-1)^{j+r}(j+r+1)!\,(x+1)}{(x+1)^{\underline{j+2+r}}} \\ &= -\frac{(i+j+r+1)!}{(i+j+r-x)^{\underline{i}}\,(-x+j+r)^{\underline{j+r+1}}} \\ &= -\frac{(i+j+r+1)!}{(i+j+r-x)^{\underline{i+j+r+1}}} \\ &= -\binom{i+j+r-x}{i+j+r+1}^{-1}.\end{aligned}$$

So we get

$$\sum_{0\le d\le \min\{i,j\}} L_{id}U_{dj} = M_{kj},$$

as claimed.

For A and A^{-1}, we have

$$\begin{aligned}\sum_{j\le d\le i} A_{id}A^{-1}_{dj} = p^{2j-2k}\frac{(-1)^j\,(N-1-j)!}{(N-1-i)!}\sum_{j\le d\le i}(-1)^d\,\frac{(2d+2+r)\,(d+j+1+r)!}{(i+d+2+r)!} \\ \times\binom{x-N-d-r}{i-d}\binom{x-N-j-r}{d-j},\end{aligned}$$

which equals 0 provided that $i\neq j$. If $i=j$, it is obvious that $A_{ii}A^{-1}_{ii}=1$. Thus

$$\sum_{i\le d\le j} A_{id}A^{-1}_{dj} = \delta_{i,j},$$

as claimed.

For B and B^{-1}, by using the Zeilberger algorithm, similarly we obtain

$$\sum_{i\le d\le j} B_{id}B^{-1}_{dj} = \delta_{i,j}.$$

For the LU-decomposition of M^{-1}, we should prove that $M^{-1}=AB$ which is same as $M=B^{-1}A^{-1}$. So it is sufficient to show that

$$\sum_{\max\{i,j\}\le d\le n-1} B^{-1}_{id}A^{-1}_{dj} = M_{ij}.$$

After some rearrangements, we have

$$\begin{aligned}\sum_{j\le d\le n-1} B^{-1}_{id}A^{-1}_{dj} = p^{2i+2j+2r+1}\sum_{j\le d\le n-1}(-1)^{d-j}\binom{x-N-j-r}{d-j}\binom{N-1+i+r-x}{i+d+2+r}^{-1} \\ \times\frac{(x+1)\,(2d+2+r)\,(N-1-i)!\,(d+j+1+r)!\,(N-1-j)!}{(N+1+d+r)!\,(d-i)!\,(N-1-d)!\,(i+d+2+r)}.\end{aligned}$$

Here we replace $(N-1)$ with N and denote the RHS of the sum by $\texttt{SUM}_N$. The Zeilberger algorithm produces the recursion

$$\texttt{SUM}_N = \texttt{SUM}_{N-1}.$$

So

$$\begin{aligned}\texttt{SUM}_N = \texttt{SUM}_j &= \frac{(x+1)\,(i+j+1+r)!}{(j+i+r-x)^{\underline{i+j+2+r}}}\\ &= \frac{(x+1)\,(i+j+1+r)!}{(j+i+r-x)^{\underline{i+j+1+r}}\,(-x-1)} = -\binom{i+j+r-x}{i+j+r+1}^{-1}\end{aligned}$$

which completes the proof.

7. Proofs for the results of Section 5

For L and L^{-1}, we have

$$\begin{aligned}\sum_{j\le d\le i} L_{id}L^{-1}_{dj} = p^{2i-2j}\frac{i!}{j!}\sum_{j\le d\le i}&\binom{i+r-x}{i-d}\binom{x-j-1-r}{d-j}\\ &\times\frac{(2d+r+s)\,(j+d+r+s-1)!}{(i+d+r+s)!}\end{aligned}$$

By the Zeilberger algorithm, the sum of the RHS of the equation just above is equal to 0 when $i\neq j$. The case $i=j$ can be easily computed. So

$$\sum_{j\le d\le i} L_{id}L^{-1}_{dj} = \delta_{i,j},$$

as desired.

For U and U^{-1},

$$\begin{aligned}\sum_{i\le d\le j} U_{id}U^{-1}_{dj} = &\frac{(-1)^{r+1}\,p^{2i-2j}x^{\underline{i+1+r}}\,(i+r+s-1)!\,(2j+r+s)!}{(2k-1+r+s)!\,(j-1+r+s)!}\\ &\times\sum_{i\le d\le j}(-1)^d\binom{-x-1}{d-1+s}\binom{-x+j+r}{d+j+r+s}\\ &\times\frac{(d-1-s)!}{(d+j+r+s)\,(d-i)!\,(j-d)!\,(i+d+r+s)!},\end{aligned}$$

which equals 0 provided that $(j-i)\,(j+i+r+s)\neq 0$. Since r and s are nonnegative integer parameters, only the case $j=i$ should be examined. Consider

$$U_{ii}U^{-1}_{ii} = \frac{(-1)^{r+1+i}\,x^{\underline{i+1+r}}\,(-x-1)^{\underline{i+s-1}}}{(-x+i+r)^{\underline{2i+r+s}}} = 1,$$

which completes the proof.

Similarly, for LU-decomposition, we have to prove that

$$\sum_{0\le d\le\min\{i,j\}} L_{id}U_{dj} = M_{ij}.$$

So without loss of generality we may choose $i \le j$. Then we obtain

$$\sum_{0\le d\le i} L_{id}U_{dj} = p^{2(i+j+r+s)}(-1)^{j+1+r+s}\sum_{0\le d\le i}\binom{-x+i+r}{i-d}\binom{x}{d+1+r}$$
$$\times\binom{-x-1}{j+s-1}\frac{i!j!(j-1+s)!(2d+r+s)(d+r+s-1)!(d+1+r)!}{d!(i+d+r+s)(j-d)!(d+j+r+s)!}.$$

Denote the above sum on the RHS in the equation above by $\mathtt{SUM}_i$, the Zeilberger algorithm produces the recurence relation for $\mathtt{SUM}_i$:

$$\mathtt{SUM}_i = \frac{i+r-x}{i+j+r+s}\mathtt{SUM}_{i-1},$$

with the initial $\mathtt{SUM}_0 = \binom{x}{1+r}\binom{-x-1}{j+s-1}\frac{(r+1)!(j-1+s)!}{(j+r+s)!}$. If we solve the recurence, we obtain

$$\mathtt{SUM}_i = \frac{(i+r-x)^{\underline{i}}}{(i+j+r+s)^{\underline{i}}}\mathtt{SUM}_0 = (-1)^{1+r}\binom{i+r-x}{i+r+j+s},$$

which completed the proof.

For A and A^{-1}, consider

$$\sum_{j\le d\le i} A_{id}A^{-1}_{dj} = p^{2j-2k}\frac{(N+i+r+s-1)!(N-1-j)!}{(N+j+r+s-1)!(N-1-i)!}$$
$$\times\sum_{j\le d\le i}\binom{x+d-1+s}{d-j}^{-1}\binom{-x-s-d}{i-d}^{-1}\frac{1}{((d-j)!)^2((i-d)!)^2}.$$

When $j \ne i$, the Zeilberger algorithm evaluates that the sum on the RHS in the equation above is equal to 0. The case $j = i$ can be easily computed.

Similarly, we have

$$\sum_{i\le d\le j} B_{id}B^{-1}_{dj} = p^{2j-2k}(-1)^k\frac{(-x)^{\underline{j+s}}}{(-x)^{\underline{i+s}}}\sum_{i\le d\le j}\frac{(-1)^d}{(j-d)!(d-k)!} = \delta_{i,j}.$$

Finally, by using the same argument in previous section, we should show that

$$\sum_{\max\{i,j\}\le d\le N-1} B^{-1}_{id}A^{-1}_{dj} = M_{ij}.$$

Consider $j \ge i$, then we have

$$\sum_{j\le d\le N-1} B^{-1}_{id}A^{-1}_{dj}$$
$$= p^{2(i+j+r+s)}(x-1)^{\underline{i+r}}(-1)^{k+r+s}\sum_{j\le d\le N-1}(-1)^{N+1}\binom{-x}{d+s}\binom{x+d-1+s}{d-j}^{-1}$$
$$\times\frac{(d+s)!(N-1-i)!(N-1-j)!(N-1+d+r+s)!}{((d-j)!)^2(d-i)!(N-1+i+r+s)!(N-1-d)!(N-1+j+r+s)!}.$$

By replacing $(N-1)$ with N on the RHS in the equation just above, we obtain the sum

$$\sum_{j\le d\le N}(-1)^N\binom{-x}{d+s}\binom{x+d-1+s}{d-j}^{-1}$$
$$\times\frac{(d+s)!\,(N-i)!\,(N-j)!\,(N+d+r+s)!}{((d-j)!)^2\,(d-i)!\,(N+i+r+s)!\,(N-d)!\,(N+j+r+s)!}.$$

Denote this sum by $\mathtt{SUM}_N$. By using the Zeilberger algorithm, we get

$$\mathtt{SUM}_N=\mathtt{SUM}_{N-1}=\mathtt{SUM}_j=\frac{(-1)^j\,(-x)^{\underline{j+s}}}{(j+i+r+s)!}.$$

Summarizing,

$$\sum_{j\le d\le N-1}B^{-1}_{id}A^{-1}_{dj}=p^{2(i+j+r+s)}\,(x-1)^{\underline{i+r}}\,(-1)^{i+r+s}\,\frac{(-1)^j\,(-x)^{\underline{j+s}}}{(j+i+r+s)!}$$
$$=(-1)^{j+s}p^{2(i+r+j+s)}\binom{i+r-x}{i+r+j+s},$$

as claimed.

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