A WILF CLASS COMPOSED OF 19 SYMMETRY CLASSES OF QUADRUPLES OF 4-LETTER PATTERNS

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ABSTRACT. In this paper, we make a contribution to the enumeration of permutations avoiding a quadruples of 4-letter patterns by establishing a Wilf class composed of 19 symmetry classes.

Keywords: pattern avoidance, Wilf-equivalence

1. INTRODUCTION

Recently, the subjects of pattern avoidance and permutation patterns have taken a lot of interest. Initially, the idea on the subjects has initiated by Knuth [3] and Simion and Schmidt [7] who considered the problem on permutations and enumerated the number of permutations of $[n] = \{1, 2, \ldots, n\}$ avoiding a particular element or subset, respectively, of patterns of length three.

Afterwards the problem has been linked with several other discrete structures, such as compositions, k-ary words, and set partitions; see, e.g., the texts [4, 5] and references contained therein.

We say a permutation is *standard* if its support set is an initial segment of the positive integers, and for a permutation π whose support is any set of positive integers, $\mathrm{St}(\pi)$ is the standard permutation obtained by replacing the smallest entry of π by 1, next smallest by 2, and so on. As usual, a standard permutation π of $[n]$ *avoids* a standard permutation τ of $[k]$ if there is no subsequence ρ of π for which $\mathrm{St}(\rho) = \tau$. In this context, τ is a *k-letter pattern* (or just simply, a *pattern*). For a set T of patterns, $S_n(T)$ denotes the set of permutations of $[n]$ that avoid all the patterns in T (for example, see [6] for subsets of at least ten 4-letter patterns and [2] for a class of three 4-letter patterns). We denote generating function for the number of permutations of $S_n(T)$ (T-avoiders of $[n]$) by $F_T(t)$, namely

$$F_T(t) = \sum_{n\geq 0} |S_n(T)|t^n.$$

Two sets of patterns T and T' are said to be *Wilf-equivalent* if their avoiders have the same counting number, that is, if $|S_n(T)| = |S_n(T')|$ for all $n \geq 0$. In the context of pattern avoidance, a *symmetry class* refers to an orbit of the dihedral group of order eight generated by the operations reverse, complement, and inverse acting entrywise on sets of patterns. Two pattern sets in the same symmetry class are trivially Wilf-equivalent.

In [1], the authors interested on the number of data structures of a certain wreath product type. In particular, they showed that the generating function of the number of permutations of $[n]$ that avoid $A_1 = \{2413, 3142, 3124, 1423\}$ is given by

$$F_{A_1}(t) = \frac{C(t)}{1 - t^3C^5(t)},$$

where

$$C(t) = \frac{1-\sqrt{1-4t}}{2t} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} t^n$$

is the generating function of the Catalan numbers.

In this paper we are interested in avoidance problem on permutations. Our experiments with numerical computations show that at most 19 classes of quadruples of 4-letter patterns have same number of avoiders with A_1-avoiders. Our main result can be formulated as follows.

Theorem 1. *Let*

$$\begin{array}{lll}
A_1 = \{2413, 3142, 3124, 1423\}, & A_2 = \{2431, 4132, 1432, 1324\}, & A_3 = \{2143, 2413, 3241, 3142\},\\
A_4 = \{2431, 2341, 2314, 3142\}, & A_5 = \{2431, 2314, 3241, 3142\}, & A_6 = \{2143, 1432, 1324, 1423\},\\
A_7 = \{2143, 1324, 1342, 1243\}, & A_8 = \{2134, 1324, 1342, 1243\}, & A_9 = \{2134, 1342, 1243, 1234\},\\
A_{10} = \{3142, 1342, 2431, 2341\}, & A_{11} = \{2431, 2413, 2314, 3241\}, & A_{12} = \{1342, 2341, 2413, 2431\},\\
A_{13} = \{3142, 1432, 1342, 1324\}, & A_{14} = \{3142, 1342, 1324, 1423\}, & A_{15} = \{3124, 1342, 1324, 1243\},\\
A_{16} = \{3124, 1324, 1423, 1243\}, & A_{17} = \{3124, 1324, 1423, 1234\}, & A_{18} = \{1324, 2341, 3241, 3421\},\\
A_{19} = \{1243, 1342, 1423, 2341\}. & &
\end{array}$$

Then the generating function for the number of permutations of size n that avoid A_j is given by

$$F_{A_j}(t) = \frac{C(t)}{1 - t^3C^5(t)},$$

where $j = 1, 2, \ldots, 19$.

To prove our main result, our main strategy is to examine the structure of an avoider, usually by splitting the class of avoiders under consideration into subclasses according to a judicious choice of parameters which may involve, for example, left-right maxima, initial letters, positions of given letters, and whether resulting subpermutations are empty or not.

2. Proofs

A permutation π expressed as $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)}$, where $i_1 < i_2 < \cdots < i_m$ and $i_j > \max(\pi^{(j)})$ for $1 \leq j \leq m$ is said to have m *left-right maxima* (at $i_1, i_2, \ldots, i_m$). Given nonempty sets of numbers S and T, we will write $S < T$ to mean $\max(S) < \min(T)$ (with the inequality vacuously holding if S or T is empty). In this context, we will often denote singleton sets simply by the element in question.

Let T be any pattern set. Throughout the paper, $a(n)$ denotes the the number of T-avoiders of size n. Moreover, $a(n; i_1i_2\cdots i_s)$ denotes the number of T-avoiders $\pi_1\pi_2\cdots\pi_n$ of size n such that $\pi_j = i_j$ for all $j = 1, 2, \ldots, s$.

2.1. Case 2: $T = \{2431, 4132, 1432, 1324\}$.

Lemma 2. *Let $T = \{2431, 4132, 1432, 1324\}$. Let $H_m(t)$ be the generating function for the number of permutations $(n+1-m)\pi^{(1)}(n+2-m)\pi^{(2)}\cdots n\pi^{(m)} \in S_n(T)$. Then $H_m(t)$ satisfies*

$$H_m(t) = t^m + \sum_{j=1}^{m-1} t^j C(t)H_{m+1-j}(t) + tH_m(t) + t^{m+1}C(t)(C(t)-1),$$

for all $m \geq 2$.

Proof. Let us write an equation for $H_m(t)$ with $m \geq 2$. Let $\pi = (n+1-m)\pi^{(1)}(n+2-m)\pi^{(2)}\cdots n\pi^{(m)} \in S_n(T)$ with $n > m$. Note that $\pi^{(1)}\pi^{(2)}\cdots\pi^{(m)}$ avoids 132. If $n-m$ belongs to $\pi^{(s)}$ with $s = 1, 2, \ldots, m-1$, then $\pi^{(1)} = \cdots = \pi^{(s-1)} = \emptyset$ because π avoids 1324, so we have a contribution of $t^s C(t)H_{m+1-s}(t)$. Otherwise, $n-m$ belongs to $\pi^{(m)}$, so we can write $\pi^{(m)} = \alpha(n-m)\beta$. If $\beta = \emptyset$, then we have a contribution of $tH_m(t)$, otherwise, $\pi = (n+1-m)\cdots n\alpha(n-m)\beta$ such that $\alpha > \beta$ and α, β avoid 132. Thus, we have a contribution of $t^{m+1}C(t)(C(t)-1)$ (see [3]). Hence,

$$H_m(t) = t^m + \sum_{j=1}^{m-1} t^j C(t)H_{m+1-j}(t) + tH_m(t) + t^{m+1}C(t)(C(t)-1),$$

where t^m counts the case $n = m$. □

Lemma 3. *Let $T = \{2431, 4132, 1432, 1324\}$. Let $J_{m,k}(t)$ be the generating function for the number of permutations*

$$(n-m)\pi^{(1)}\cdots(n+k-1-m)\pi^{(m-k)}(n+k+1-m)\pi^{(m+1-k)}\cdots n\pi^{(m)}(n+k-m) \in S_n(T).$$

Then $J_{m,1}(t) = tH_m(t)$ and $J_{m,k}(t) = t^k C^{k-1}(t)H_{m+1-k}$ for all $k = 2, 3, \ldots, m-1$, where $H_m(t)$ is given in statement of Lemma 2.

Proof. Let us find a formula for $J_{m,k}(t)$. Let $\pi \in S_n(T)$ with the form

$$\pi = (n-m)\pi^{(1)}\cdots(n+k-1-m)\pi^{(m-k)}(n+k+1-m)\pi^{(m+1-k)}\cdots n\pi^{(m)}(n+k-m).$$

By Lemma 2, we have that $J_{m,1}(t) = tH_m(t)$. Let $k = 2, 3, \ldots, m-1$. Since π avoids 1324, we see that $\pi^{(1)} > \pi^{(2)} > \cdots > \pi^{(k-1)} > \pi^{(k)}\pi^{(k+1)}\cdots\pi^{(m)}$ and $\pi^{(s)}$ avoids 132 for all $s = 1, 2, \ldots, k-1$, which, by Lemma 2, implies that $J_{m,k}(t) = t^k C^{k-1}(t)H_{m+1-k}(t)$, as required. □

Theorem 4. *Let $T = \{2431, 4132, 1432, 1324\}$. The generating function for T-avoiders is given by*

$$F_T(t) = \frac{C(t)}{1 - t^3C^5(t)}.$$

Proof. Let $G_m(t)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(t) = 1$ and $G_1(t) = tC(t)$.

Let us write an equation for $G_m(t)$ with $m \geq 2$. Let $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima such that $i_1 < i_2 < \cdots < i_m = n$. Since π avoids 1324, we see that $\pi^{(j)} < i_1$ for all $j = 1, 2, \ldots, m-1$. If $\pi^{(m)} < i_1$ then we have a contribution of $H_m(t)$ (as discussed in Lemma 2). Otherwise, $\pi^{(m)}$ has a letter between i_j and i_{j+1} for

$1 \le j \le m-1$. Since π avoids T, we see that $\pi^{(m)} = \alpha(i_j+1)(i_j+2)\cdots(i_{j+1}-1)$ with $\alpha < i_1$. Hence, by Lemma 3, we have a contribution of $J_{m,j}(t)$. Hence,

$$G_m(t) = H_m(t) + \frac{t}{1-t}H_m(t) + \sum_{j=2}^{m-1}\frac{t^jC^{j-1}(t)}{1-t}H_{m+1-j}(t).$$

Thus, by summing over $m \ge 2$, we have that

$$\sum_{m\ge2} G_m(t) = \left(\frac{1}{1-t} + \frac{t^2C(t)}{(1-t)(1-tC(t))}\right)\sum_{m\ge2} H_m(t),$$

which implies

$$F_T(t) = 1 + tC(t) + \frac{1+t^2C^2(t)}{1-t}\sum_{m\ge2} H_m(t).$$

On the other hand, by Lemma 2, we have

$$\sum_{m\ge2} H_m(t) = \frac{t^2(1+tC(t)(C(t)-1))}{(1-t)^2-tC(t)}.$$

Hence, by after several algebraic operations with using the fact that $C(t) = 1 + tC^2(t)$, we obtain that $F_T(t) = \frac{C(t)}{1-t^3C^5(t)}$, as required. □

2.2. **Case 3: $T = \{2143, 2413, 3241, 3142\}$**.

Theorem 5. *Let $T = \{2143, 2413, 3241, 3142\}$. The generating function for T-avoiders is given by*

$$F_T(t) = \frac{C(t)}{1-t^3C^5(t)}.$$

Proof. Let $G_m(t)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(t) = 1$ and $G_1(t) = tF_T(t)$.

Let us write an equation for $G_m(t)$ for all $m \ge 2$. Let $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima such that $i_1 < i_2 < \cdots < i_m = n$. If $\pi^{(j)} > i_1$ for all $j = 1, 2, \ldots, m$ (that is, $i_1 = 1$), then we have a contribution of $tG_{m-1}(t)$. Otherwise, there exists s, $1 \le s \le m$, such that $\pi^{(s)}$ has a letter smaller than i_1. Since π avoids 3241 and 3142, we see that $\pi^{(j)} > i_1$ for all $j = 1, 2, \ldots, s-1, s+1, s+2, \ldots, m$. Since π avoids T, we see that $\pi^{(j)} = \emptyset$ for all $j \neq s$. Since π avoids 2413, we can write $\pi^{(s)}$ as $\alpha^{(s)}\alpha^{(s-1)}\cdots\alpha^{(1)}$ such that $i_{j-1} < \alpha^{(j)} < i_j$ for all $j = 2, \ldots, s$ and $\alpha^{(1)} \neq \emptyset$ such that $\alpha^{(1)} < i_1$. Note that π avoids T if and only if $\alpha^{(j)}$ avoids 213 for all $j = s, s-1, \ldots, 2$ and $\alpha^{(1)}$ avoids T. Also, the generating function for the number of $\{213\}$-avoiders is given by $C(t)$, see [3]. Thus, we have a contribution of $t^mC^{s-1}(t)(F_T(t)-1)$. Hence,

$$G_m(t) = tG_{m-1}(t) + t^m(F_T(t)-1)\sum_{s=1}^{m} C^{s-1}(t).$$

By summing over $m \ge 2$, we obtain

$$F_T(t) - 1 - tF_T(t) = t(F_T(t)-1) + \frac{t^2(F_T(t)-1)}{(1-t)(1-C(t))} - \frac{t^2C^3(t)(F_T(t)-1)}{1-C(t)},$$

which, by using the fact that $C(t)=1+tC^2(t)$, leads to $F_T(t)=\frac{C(t)}{1-t^3C^5(t)}$, as claimed. □

2.3. Case 4: $T=\{2431,2314,2341,3142\}$.

Theorem 6. *Let* $T=\{2431,2314,2341,3142\}$. *The generating function for* T*-avoiders is given by*

$$F_T(t)=\frac{C(t)}{1-t^3C^5(t)}.$$

Proof. Let $G_m(t)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(t)=1$ and $G_1(t)=tF_T(t)$.

Let us write an equation for $G_2(t)$. Let $\pi=i\pi'n\pi''\in S_n(T)$ with exactly 2 left-right maxima. If $\pi''>i$ then π' and π'' must avoid 231 and T, respectively. So we have a contribution of $t^2C(t)F_T(t)$. Otherwise, π'' has a letter smaller than i. By considering if $\pi''<i$ (that is $i=n-1$) or not, we obtain the contributions of $\frac{t^2}{1-t}(F_T(t)-1)$ and $\frac{t^2}{1-t}(C(t)-1)(F_T(t)-1)$, respectively (we used the fact that the generating function for the number of 231-avoiders is given by $C(t)$, see [3]). Hence,

$$G_2(t)=t^2C(t)F_T(t)+\frac{t^2}{1-t}(F_T(t)-1)+\frac{t^2}{1-t}(C(t)-1)(F_T(t)-1).$$

Let us write an equation for $G_m(t)$ for all $m\geq 3$. Let $\pi=i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)}\in S_n(T)$ with exactly m left-right maxima such that $i_1<i_2<\cdots<i_m=n$. Since π avoids 2314 and 2341, we see that $\pi^{(j)}>i_1$ for all $j=2,3,\ldots,m$. Thus, we have $G_m(t)=tC(t)G_{m-1}(t)$. By summing over $m\geq 3$, we obtain

$$F_T(t)-1-tF_T(t)-G_2(t)=tC(t)(F_T(t)-1-tF_T(t)),$$

which, by solving and using the fact that $C(t)=1+tC^2(t)$, implies that $F_T(t)=\frac{C(t)}{1-t^3C^5(t)}$, as claimed. □

2.4. Case 5: $T=\{2431,2314,3241,3142\}$.

Theorem 7. *Let* $T=\{2431,2314,3241,3142\}$. *The generating function for* T*-avoiders is given by*

$$F_T(t)=\frac{C(t)}{1-t^3C^5(t)}.$$

Proof. Let $G_m(t)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(t)=1$ and $G_1(t)=tF_T(t)$.

Let us write an equation for $G_2(t)$. Let $\pi=i\pi'n\pi''\in S_n(T)$ with exactly 2 left-right maxima. Since π avoids 2431 and 3142 we can write π as $i\pi'n\alpha\beta$ such that $\alpha<\pi'<i<\beta$. By considering either α is empty or not (in this case π' is empty), we have the contributions of $t^2C(t)F_T(t)$ and $t^2C(t)(F_T(t)-1)$, respectively. Hence,

$$G_2(t)=t^2C(t)F_T(t)+t^2C(t)(F_T(t)-1).$$

Let us write an equation for $G_m(t)$ for all $m\geq 3$. Let $\pi=i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)}\in S_n(T)$ with exactly m left-right maxima such that $i_1<i_2<\cdots<i_m=n$. Since π avoids 2314,

we see that $\pi^{(j)} > i_1$ for all $j = 2, 3, \ldots, m-1$. If $\pi^{(m)} > i_1$ then we have a contribution of $tC(t)G_{m-1}(t)$. Otherwise, $\pi^{(m)}$ has a letter smaller than i_1, so, since π avoids T we can write π as $\pi = i_1 i_2 \cdots i_m \alpha\beta$ such that $\alpha < i_1 < \beta < i_2$. By considering either β is empty or not, we obtain the contributions $t^m(F_T(t)-1)$ and $t^m(C(t)-1)(F_T(t)-1)$. Hence,

$$G_m(t) = tC(t)G_{m-1}(t) + t^m C(t)(F_T(t)-1).$$

By summing over $m \geq 3$, we obtain

$$F_T(t) - 1 - tF_T(t) - G_2(t) = tC(t)(F_T(t) - 1 - tF_T(t)) + \frac{t^3}{1-t}C(t)(F_T(t)-1),$$

which, by solving and using the fact that $C(t) = 1 + tC^2(t)$, implies that $F_T(t) = \frac{C(t)}{1-t^3C^5(t)}$, as claimed. □

2.5. **Case 6: $T = \{2143, 1324, 1342, 1432\}$.**

Theorem 8. *Let $T = \{2143, 1324, 1342, 1432\}$. The generating function for T-avoiders is given by*

$$F_T(t) = \frac{C(t)}{1-t^3C^5(t)}.$$

Proof. Let $G_m(t)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(t) = 1$ and $G_1(t) = tF_T(t)$.

Let us write an equation for $G_2(t)$. Let $\pi = i\pi' n\pi'' \in S_n(T)$ with exactly 2 left-right maxima. If $\pi'' < i$ then we have a contribution of $t(F_T(t)-1)$. Otherwise, π'' has a letter greater than i, so $\pi' = \emptyset$ and π'' includes the subsequence $(i+1)(i+2)\cdots(n-1)$. Since π avoids T, then $\pi''^{(1)}(i+1)\alpha^{(2)}(i+2)\cdots\alpha^{(n-i-1)}(n-1)\alpha^{(n-i)}$ such that $\alpha^{(1)} > \alpha^{(2)} > \cdots > \alpha^{(n-i-2)} > \alpha^{(n-i-1)} > \alpha^{(n-i)}$. Thus, we have a contribution of $\frac{t^2(F_T(t)-1)}{1-tC(t)}$. Hence,

$$G_2(t) = t(F_T(t)-1) + \frac{t^2(F_T(t)-1)}{1-tC(t)}.$$

Let us write an equation for $G_m(t)$ for all $m \geq 3$. Let $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima such that $i_1 < i_2 < \cdots < i_m = n$. Since π avoids 2143, we see that $\pi^{(j)} < i_1$ for all $j = 1, 2, \ldots, m-1$. Since π avoids 1324 and 1342, we see that $\pi^{(1)} > \pi^{(j)}$ for all $j = 2, 3, \ldots, m-1$ and $\pi^{(m)}$ has no letter between i_1 and i_{m-1}. Hence, by case $m = 2$ and considering either $\pi^{(1)}$ empty or not, we have that

$$G_m(t) = tG_{m-1}(t) + t(C(t)-1)(tC(t))^{m-2}t(F_T(t)-1).$$

By summing over $m \geq 3$, we obtain

$$F_T(t) - 1 - tF_T(t) - G_2(t) = t(F_T(t) - 1 - tF_T(t)) + +t(C(t)-1)t(F_T(t)-1)/(1-tC(t)),$$

which, by solving and using the fact that $C(t) = 1 + tC^2(t)$, implies that $F_T(t) = \frac{C(t)}{1-t^3C^5(t)}$, as claimed. □

2.6. **Case 7: $T = \{2143, 1324, 1342, 1243\}$.**

Theorem 9. *Let $T = \{2143, 1324, 1342, 1243\}$. The generating function for T-avoiders is given by*

$$F_T(t) = \frac{C(t)}{1 - t^3C^5(t)}.$$

Proof. Let $G_m(t)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(t) = 1$ and $G_1(t) = tF_T(t)$.

Let us write an equation for $G_2(t)$. Let $\pi = i\pi' n\pi'' \in S_n(T)$ with exactly 2 left-right maxima. If $\pi'' < i$ then we have a contribution of $t(F_T(t) - 1)$. Otherwise, π'' has a letter greater than i, so $\pi' = \emptyset$, which implies a contribution of $t(F_T(t) - 1 - tF_T(t))$. Hence,

$$G_2(t) = t(F_T(t) - 1) + t(F_T(t) - 1 - tF_T(t)).$$

Let us write an equation for $G_m(t)$ for all $m \geq 3$. Let $\pi = i_1\pi^{(1)}i_2\pi^{(2)} \cdots i_m\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima such that $i_1 < i_2 < \cdots < i_m = n$. Since π avoids 1324, 1342 and 1243, we see that $\pi^{(j)} < i_1$ for all $j = 1, 2, \ldots, m$. Since π avoids 1324 and 1342, we see that $\pi^{(1)} > \pi^{(2)} > \cdots > \pi^{(m-2)} > \pi^{(m-1)}\pi^{(m)}$. Hence, by case $m = 2$, we have that $G_m(t) = (tC(t))^{m-2}t(F_T(t) - 1)$. By summing over $m \geq 3$, we obtain

$$F_T(t) - 1 - tF_T(t) - G_2(t) = t^2C(t)(F_T(t) - 1)/(1 - tC(t)),$$

which, by solving and using the fact that $C(t) = 1 + tC^2(t)$, implies that $F_T(t) = \frac{C(t)}{1-t^3C^5(t)}$, as claimed. ◻

2.7. **Case 8: $T = \{1243, 1324, 1342, 2134\}$.**

Lemma 10. *For all $1 \leq i \leq n - 3$,*

$$a(n; i) = a(n - 1; i) + \sum_{j=1}^{i} a(n - 1; j) - \sum_{j=1}^{i} a(n - 2; j) - \sum_{j=1}^{i-1} a(j)$$

with $a(n; n - 2) = a(n - 1) - \sum_{j=1}^{n-3} a(j)$ and $a(n; n) = a(n; n - 1) = a(n - 1)$.

Proof. The initial conditions $a(n; n) = a(n; n - 1) = a(n - 1)$ and $a(n; i, n) = a(n - 1; i)$ with $1 \leq i \leq n - 1$ easily follow from the definitions. For $1 \leq j < i \leq n - 2$, since π avoids 2134, we have

$$a(n; i, j) = \sum_{k=1}^{j-1} a(n; i, j, k) + a(n; i, j, n) = \sum_{k=1}^{j-1} a(n - 1; j, k) + a(n - 1; i, j).$$

Since π avoids 1324 and 1342, we have that $a(n; i, j) = 0$ for all $2 \leq i + 1 < j \leq n - 1$. Since π avoids T, we see that $a(n; i, i+1) = a(n; i, i+1, i+2, \ldots, n-1) = a(i)$. Therefore, for $1 \leq j < i \leq n - 2$,

$$\begin{aligned} a(n; i, j) &= a(n - 1; j) - a(n - 1; j, j + 1) - a(n - 1; j, n) + a(n - 1; i, j) \\ &= a(n - 1; j) - a(j) - a(n - 2; j) + a(n - 1; i, j), \end{aligned}$$

which, by summing over $j = 1, 2, \dots, i-1$, implies

$$\begin{aligned} a(n;i) - a(n-1;i) - a(i) &= \sum_{j=1}^{i-1} a(n-1;j) - \sum_{j=1}^{i-1} a(n-2;j) - \sum_{j=1}^{i-1} a(j) \\ &\quad + a(n-1;i) - a(n-2;i) - a(i). \end{aligned}$$

Thus, for all $i = 1, 2, \dots, n-3$,

$$a(n;i) = a(n-1;i) + \sum_{j=1}^{i} a(n-1;j) - \sum_{j=1}^{i} a(n-2;j) - \sum_{j=1}^{i-1} a(j)$$

with

$$\begin{aligned} a(n;n-2) &= a(n-1;n-2) + \sum_{j=1}^{n-1} a(n-1;j) - \sum_{j=1}^{n-2} a(n-2;j) - \sum_{j=1}^{n-3} a(j) \\ &= a(n-1) - \sum_{j=1}^{n-3} a(j) \end{aligned}$$

with $a(n;n) = a(n;n-1) = a(n-1)$, as claimed. □

Theorem 11. *Let $T = \{1243, 1324, 1342, 2134\}$. The generating function for T-avoiders is given by*

$$F_T(t) = \frac{C(t)}{1 - t^3 C^5(t)}.$$

Proof. Define $A_n(v) = \sum_{i=1}^{n} a(n;i) v^{i-1}$. Then the recurrence in Lemma 10can be written as

$$\begin{aligned} A_n(v) &= A_{n-1}(1)(v^{n-1} + v^{n-2} + v^{n-3}) + A_{n-1}(v) - A_{n-2}(1)(v^{n-2} + v^{n-3}) \\ &\quad + \frac{A_{n-1}(v) - v^{n-2} A_{n-2}(1) - v^{n-3}(A_{n-1}(1) - A_{n-2}(1))}{1-v} \\ &\quad - \frac{A_{n-2}(v) - v^{n-3} A_{n-2}(1)}{1-v} - \sum_{j=1}^{n-2} \frac{v^j - v^{n-2}}{1-v} a(j), \end{aligned}$$

with initial conditions $A_0(v) = A_1(v) = 1$ and $A_2(v) = 1 + v$.

Define $A(t;v) = \sum_{n \geq 0} A_n(v) t^n$. Hence,

$$\begin{aligned} A(t;v) &= 1 + t + (1+v)t^2 + t(1 + 1/v + 1/v^2)(A(tv;1) - 1 - vt) \\ &\quad + t(A(t;v) - 1 - t) - t^2(1 + 1/v)A(tv;1) \\ &\quad + \frac{t(A(t;v) - 1 - t) - t^2(A(tv;1) - 1) - \frac{t}{v^2}(A(tv;1) - 1 - vt) + \frac{t^2}{v}(A(tv;1) - 1)}{1-v} \\ &\quad - \frac{t^2(A(t;v) - 1) - \frac{t^2}{v}(A(tv;1) - 1)}{1-v} \\ &\quad - \frac{t^3}{(1-t)(1-v)}(A(tv;1) - 1) + \frac{t^3 v}{(1-vt)(1-v)}(A(tv;1) - 1), \end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{(t-v+v^2)(t-v)}{(1-v)v^2}A(t/v;v)\\
&=\frac{t(-v^4+v(v^3+2v^2+1-v)t-(1+2v)(v^2-v+1)t^2+(v^2-v+1)t^3)}{v^3(1-v)(v-t)(1-t)}A(t;1)\\
&+\frac{-v^4+v^3(v+2)t-v(1+v)(2v-1)t^2+(v^2-2v-1)t^3+t^4}{v^3(1-t)(t-v)}.
\end{aligned}$$

To solve the preceding functional equation, we apply the kernel method and take $v=\frac{1+\sqrt{1-4t}}{2}=1/C(t)$. Then after several algebraic operations with using $C(t)=1+tC^2(t)$, we obtain that $A(t;1)=\frac{C(t)}{1-t^3C^5(t)}$, as claimed. □

2.8. Case 9: $T=\{1234,1243,1342,2134\}$.

Lemma 12. *For all* $1\leq i\leq n-3$,

$$a(n;i)=a(n-1;i)+\sum_{j=1}^{i}a(n-1;j)-\sum_{j=1}^{i}a(n-2;j)-\sum_{j=1}^{i-1}a(j)$$

with $a(n;n-2)=a(n-1)-\sum_{j=1}^{n-3}a(j)$ *and* $a(n;n)=a(n;n-1)=a(n-1)$.

Proof. The initial conditions $a(n;n)=a(n;n-1)=a(n-1)$ and $a(n;i,n)=a(n-1;i)$ with $1\leq i\leq n-1$ easily follow from the definitions. For $1\leq j<i\leq n-2$, since π avoids 2134, we have

$$a(n;i,j)=\sum_{k=1}^{j-1}a(n;i,j,k)+a(n;i,j,n)=\sum_{k=1}^{j-1}a(n-1;j,k)+a(n-1;i,j).$$

Since π avoids 1234 and 1243, we have that $a(n;i,j)=0$ for all $1\leq i<j\leq n-2$. Since π avoids 1342 and 1234, we see that $a(n;i,n-1)=a(n;i,n-1,n-2,\ldots,i+1)=a(i)$. Therefore, for $1\leq j<i\leq n-2$,

$$a(n;i,j)=a(n-1;j)-a(n-1;j,n-1)-a(n-1;j,n-2)+a(n-1;i,j),$$

which, by summing over $j=1,2,\ldots,i-1$, implies

$$\begin{aligned}
a(n;i)-a(n-1;i)-a(i)=&\sum_{j=1}^{i-1}a(n-1;j)-\sum_{j=1}^{i-1}a(n-2;j)-\sum_{j=1}^{i-1}a(j)\\
&+a(n-1;i)-a(n-2;i)-a(i).
\end{aligned}$$

Thus, for all $i = 1, 2, \ldots, n-3$,

$$a(n;i) = a(n-1;i) + \sum_{j=1}^{i} a(n-1;j) - \sum_{j=1}^{i} a(n-2;j) - \sum_{j=1}^{i-1} a(j),$$

$$\begin{aligned} a(n;n-2) &= a(n-1;n-2) + \sum_{j=1}^{n-1} a(n-1;j) - \sum_{j=1}^{n-2} a(n-2;j) - \sum_{j=1}^{n-3} a(j) \\ &= a(n-1) - \sum_{j=1}^{n-3} a(j) \end{aligned}$$

with $a(n;n) = a(n;n-1) = a(n-1)$, as required. □

By Lemmas 10 and 12 with using Theorem 11, we have the following result.

Theorem 13. *Let $T = \{1234, 1243, 1342, 2134\}$. The generating function for T-avoiders is given by*

$$F_T(t) = \frac{C(t)}{1 - t^3 C^5(t)}.$$

2.9. **Case 10: $T = \{3142, 1342, 2431, 2341\}$.**

Theorem 14. *Let $T = \{3142, 1342, 2431, 2341\}$. The generating function for T-avoiders is given by*

$$F_T(t) = \frac{C(t)}{1 - t^3 C^5(t)}.$$

Proof. Let $G_m(t)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(t) = 1$ and $G_1(t) = tF_T(t)$.

Let us write an equation for $G_2(t)$. Let $\pi = i\pi' n\pi'' \in S_n(T)$ with exactly 2 left-right maxima. Since π avoids 3142 and 2431, we can write π as $i\pi' n\alpha\beta$ such that $\alpha < \pi' < i < \beta$. By considering either α is empty or not, we get the contributions $t^2 F_T(t)C(t)$ and $\frac{t^2}{1-t}C(t)(C(t)-1)$, respectively, where $C(t)$ counts the 231-avoiders.

$$G_2(t) = t^2 F_T(t)C(t) + \frac{t^2}{1-t}C(t)(C(t)-1).$$

Let us write an equation for $G_m(t)$ for all $m \geq 3$. Let $\pi = i_1\pi^{(1)} i_2\pi^{(2)} \cdots i_m\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima such that $i_1 < i_2 < \cdots < i_m = n$. Since π avoids 1342 and 2341, we see that $\pi^{(j)} > i_{j-1}$ for all $j = 3, 4, \ldots, m$. So, π avoids T if and only if $i_1\pi^{(1)} i_2\pi^{(2)}$ avoids T and $\pi^{(j)}$ avoids 231 for all $j \geq 3$. Therefore,

$$G_m(t) = t^{m-2} C^{m-2}(t) G_2(t).$$

By summing over $m \geq 2$, we obtain

$$F_T(t) - 1 - tF_T(t) = \frac{t^2 F_T(t)C(t) + \frac{t^2}{1-t}C(t)(C(t)-1)}{1 - tC(t)} = t^2 F_T(t)C^2(t) + \frac{t^3}{1-t}C^4(t),$$

which, by using the fact that $C(t) = 1 + tC^2(t)$, leads to $F_T(t) = \frac{C(t)}{1-t^3C^5(t)}$, as claimed. □

2.10. Case 11: $T = \{2431, 2413, 2314, 3241\}$.

Theorem 15. *Let* $T = \{2431, 2413, 2314, 3241\}$. *The generating function for* T*-avoiders is given by*
$$F_T(t) = \frac{C(t)}{1 - t^3C^5(t)}.$$

Proof. Let $G_m(t)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(t) = 1$ and $G_1(t) = tF_T(t)$.

Let us write an equation for $G_2(t)$. Let $\pi = i\pi' n\pi'' \in S_n(T)$ with exactly 2 left-right maxima. If π'' has a letter smaller than i, then $\pi' < \pi''$ such that π' avoids 231 and π'' avoids T. Thus, we have a contribution of $t^2C(t)(F_T(t) - 1)$. Otherwise, $\pi' < i < \pi''$, which implies a contribution of $t^2C(t)F_T(t)$. Hence,
$$G_2(t) = t^2F_T(t)C(t) + t^2C(t)(F_T(t) - 1).$$

Let us write an equation for $G_m(t)$ for all $m \geq 3$. Let $\pi = i_1\pi^{(1)}i_2\pi^{(2)} \cdots i_m\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima such that $i_1 < i_2 < \cdots < i_m = n$. Since π avoids 2314, we see that $\pi^{(j)} > i_{j-1}$ for all $j = 2, 3, \ldots, m-1$. If $\pi^{(m)} > i_1$ then we have a contribution of $tC(t)G_{m-1}(t)$. Otherwise, $\pi^{(m)}$ contains a letter smaller than i_1. Since π avoids 2431 and 2413, then we see that $\pi = i_1\pi^{(1)}i_2 \cdots i_m\pi^{(m)}$ such that $\pi^{(1)} < \pi^{(m)} < i_1$. Thus, we have a contribution of $t^mC(t)(F_T(t) - 1)$. Hence,
$$G_m(t) = tC(t)G_{m-1}(t) + t^mC(t)(F_T(t) - 1).$$
By summing over $m \geq 3$, we obtain
$$F_T(t) - 1 - tF_T(t) - G_2(t) = tC(t)(F_T(t) - 1 - tF_T(t)) + \frac{t^3}{1-t}C(t)(F_T(t) - 1),$$
which, by using the fact that $C(t) = 1 + tC^2(t)$, leads to $F_T(t) = \frac{C(t)}{1-t^3C^5(t)}$, as claimed. □

2.11. Case 12: $T = \{1342, 2341, 2413, 2431\}$.

Theorem 16. *Let* $T = \{1342, 2341, 2413, 2431\}$. *The generating function for* T*-avoiders is given by*
$$F_T(t) = \frac{C(t)}{1 - t^3C^5(t)}.$$

Proof. Let $G_m(t)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(t) = 1$ and $G_1(t) = tF_T(t)$.

Let us write an equation for $G_2(t)$. Let $\pi = i\pi' n\pi'' \in S_n(T)$ with exactly 2 left-right maxima. If $\pi'' < i$ then we have a contribution of $t(F_T(t) - 1)$. Otherwise, since π avoids 2413 and 2431, we have that $\pi'' > i$. So π avoids T if and only if π' avoids T and π'' avoids 231. Thus, by [3], the contribution is given by $t^2F_T(t)(C(t) - 1)$. Hence,
$$G_2(t) = t(F_T(t) - 1) + t^2F_T(t)(C(t) - 1) = t(F_T(t) - 1) + t^3F_T(t)C^2(t).$$

Let us write an equation for $G_m(t)$ for all $m \geq 3$. Let $\pi = i_1\pi^{(1)}i_2\pi^{(2)} \cdots i_m\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima such that $i_1 < i_2 < \cdots < i_m = n$. Since π avoids 1342 and 2341, we see that $\pi^{(j)} > i_{j-1}$ for all $j = 3, 4, \ldots, m$. So, π avoids T if and only if $i_1\pi^{(1)}i_2\pi^{(2)}$ avoids T and $\pi^{(j)}$ avoids 231 for all $j \geq 3$. Therefore,

$$G_m(t) = t^{m-2}C^{m-2}(t)G_2(t).$$

By summing over $m \geq 2$, we obtain

$$F_T(t) - 1 - tF_T(t) = \frac{t(F_T(t) - 1) + t^3F_T(t)C^2(t)}{1 - tC(t)} = t(F_T(t) - 1)C(t) + t^3F_T(t)C^3(t),$$

which, by using the fact that $C(t) = 1 + tC^2(t)$, leads to $F_T(t) = \frac{C(t)}{1-t^3C^5(t)}$, as claimed. □

2.12. Case 13: $T = \{1324, 1423, 1432, 2413\}$.

Lemma 17. *For all* $1 \leq i \leq n - 2$,

$$a(n; i) = a(n - 1; 1) + a(n - 1; 2) + \cdots + a(n - 1; i) + a(i - 1)$$

with $a(n; n) = a(n; n - 1) = a(n - 1)$.

Proof. The initial conditions $a(n; n) = a(n; n - 1) = a(n - 1)$ easily follow from the definitions. For $1 \leq i \leq n - 2$, we have

$$a(n; i) = \sum_{j=1}^{i-1} a(n; ij) + a(n; i(i + 1)) + a(n; i(i + 2)) + \sum_{j=i+3}^{n} a(n; ij).$$

So assume $1 \leq j < i \leq n - 2$ and let $\pi = ij\pi'$ be a member of $S_n(T)$. If π avoids T then $j\pi'$ avoids T. Let π contains T and $j\pi'$ avoids T, so there is an occurrence $iabc$ of 2413 in π such that $b < i < a < c$. If $b < j$ then $j\pi'$ contains 2413, otherwise $b > j$ and $j\pi'$ contains 1423, a contradiction. Thus, π avoids T if and only if $j\pi'$ avoids T, which implies $a(n; ij) = a(n - 1; j)$.

Let $\pi = i(i + 1)\pi'$ be a member of $S_n(T)$, here π avoids T if and only if $i\pi'$ avoids T, so $a(n; i(i + 1)) = a(n - 1; i)$.

Let $\pi = i(i+2)\pi'$ be a member of $S_n(T)$, then π can be written as $i(i+2)\alpha(i+1)\beta$. Since π avoids T, we see that $\pi = i(i + 2)(i + 3) \cdots n(i + 1)\beta$. Thus, π avoids T if and only if β avoids T, which implies that $a(n; i(i + 2)) = a(i - 1)$.

So assume $3 \leq i + 2 < j \leq n$ and let $\pi = ij\pi'$ be a member of $S_n(T)$, then π contains $ij(i + 1)(i + 2)$ or $ij(i + 2)(i + 1)$, which leads to $a(n; ij) = 0$.

Hence, $a(n; i) = a(n-1; 1)+a(n-1; 2)+\cdots+a(n-1; i)+a(i-1)$ for all $i = 1, 2, \ldots, n-2$, as claimed. □

Theorem 18. *Let* $T = \{1324, 1423, 1432, 2413\}$. *The generating function for* T*-avoiders is given by*

$$F_T(t) = \frac{C(t)}{1 - t^3C^5(t)}.$$

Proof. Define $A_n(v) = \sum_{i=1}^{n} a(n; i)v^{i-1}$. Lemma 17 gives

$$A_n(v) - (v^{n-1} + v^{n-2})A_{n-1}(1) = \frac{1}{1 - v}(A_{n-1}(v) - v^{n-2}A_{n-1}(1)) + \sum_{j=0}^{n-3} v^jA_j(1),$$

which is equivalent to

$$A_n(v) = \frac{1}{1-v}(A_{n-1}(v) - v^n A_{n-1}(1)) + \sum_{j=0}^{n-3} v^j A_j(1).$$

Here, we can use the initial conditions $A_0(v) = A_1(v) = 1$.

Define $A(t;v) = \sum_{n\geq 0} A_n(v)t^n$. The above recurrence relation can be written as

$$A(t;v) = 1 + \frac{t}{1-v}(A(t;v) - vA(vt;1) + \frac{t^3}{1-t}A(vt;1),$$

which is equivalent to

$$\left(1 - \frac{t}{v(1-v)}\right) A(t/v;v) = 1 + \left(\frac{t^3}{v^2(v-t)} - \frac{t}{1-v}\right) A(t;1).$$

To solve the preceding functional equation, we apply the kernel method and take $v = \frac{1+\sqrt{1-4t}}{2} = 1/C(t)$. Then $A(t;1) = \frac{C(t)}{1-t^3C^5(t)}$, as claimed. □

2.13. Case 14: $T = \{1324, 1342, 1423, 2413\}$

Lemma 19. *For all* $1 \leq i \leq n-2$,

$$a(n;i) = a(n-1;1) + a(n-1;2) + \cdots + a(n-1;i) + a(i-1)$$

with $a(n;n) = a(n;n-1) = a(n-1)$.

Proof. The initial conditions $a(n;n) = a(n;n-1) = a(n-1)$ easily follow from the definitions. For $1 \leq i \leq n-2$, we have

$$a(n;i) = \sum_{j=1}^{i-1} a(n;ij) + a(n;i(i+1)) + \sum_{j=i+2}^{n-1} a(n;ij) + a(n;in).$$

So assume $i+1 < j < n$ and let $\pi = ij\pi'$ be a member of $S_n(T)$, then π contains 1324 or 1342, which leads to $a(n;ij) = 0$. Moreover, the case $a(n;i\,(i+1)) = a(n-1;i)$ since at the beginning we have no increasingly consecutive pattern in T.

So assume $1 \leq j < i \leq n-2$ and let $\pi = ij\pi'$ be a member of $S_n(T)$. The following statement holds: π avoids T if and only if $j\pi'$ avoids T. Necessary part is clear. Assume that π contains T and $j\pi'$ avoids T, so there is an occurrence $iabc$ of 2413 in π such that $b < i < c < a$. If $b < j$ then $j\pi'$ contains 2413, otherwise $b > j$ and $jabc$ is the pattern 1423, a contradiction. Thus $a(n;ij) = a(n-1;j)$.

Let $\pi = in\pi'$ be a member of $S_n(T)$, since π avoids 1423, we have that π contains the subsequence $in\,(n-1)\;(n-2)\cdots(i+1)$. Since π avoids 2413, π must be written as $\pi = in\,(n-1)\cdots(i+1)\,\pi''$, which implies $a(n;in) = a(i-1)$.

Hence, $a(n;i) = a(n-1;1) + a(n-1;2) + \cdots + a(n-1;i) + a(i-1)$, as claimed. □

By Lemmas 17 and 19 with using Theorem 18, we obtain the following result.

Theorem 20. *Let* $T = \{1324, 1342, 1423, 2413\}$. *The generating function for* T*-avoiders is given by*

$$F_T(t) = \frac{C(t)}{1-t^3C^5(t)}.$$

2.14. Case 15: $T = \{1243, 1324, 1423, 2314\}$

Lemma 21. *Let $b(n;i) = a(n;in)$. For all $1 \le i \le n-2$,*

$$a(n;i) = \sum_{j=1}^{i} a(n-1;j) - \sum_{j=1}^{i-1} a(n-2;j) + b(n;i)$$

with $a(n;n) = a(n;n-1) = a(n-1)$, where for all $1 \le i \le n-3$,

$$b(n;i) = b(n-1;1) + \cdots + b(n-1;i)$$

with $b(n;n-1) = b(n;n-2) = a(n-2)$ and $b\,(n;n) = 0$.

Proof. The initial conditions $a(n;n) = a(n;n-1) = a(n-1)$, $b(n;n-1) = a(n;(n-1)n) = a(n-2)$, $b(n;n-2) = a(n;(n-2)n) = a(n-2)$ and $b\,(n;n) = 0$ easily follow. For $1 \le i \le n-2$, we have

$$a(n;i) = \sum_{j=1}^{i-1} a(n;ij) + \sum_{j=i+1}^{n-1} a(n;ij) + a(n;in).$$

So assume $1 \le j < i \le n-2$ and let $\pi = ij\pi'$ be a member of $S_n(T)$. If π avoids T then clearly $j\pi'$ avoids T. Let's assume that π contains T and $j\pi'$ avoids T, so there is an occurrence $iabc$ of 2314 in π such that $b < i < a < c$. If $b < j$ then $jabc$ is the pattern 2314, otherwise $b > j$ and $j\pi'$ contains 1324, a contradiction. Thus, π avoids T if and only if $j\pi'$ avoids T, which implies $a(n;ij) = a(n-1;j)$.

For $1 \le i < j \le n-1$, let $\pi = ij\pi'$ be a member of $S_n(T)$. Since $1243 \in T$, π contains the subsequence $ij(j+1)\cdots n$. Since π avoids 2314, 1324 and $j \ne n$, we have that n is located $(n-j+2)$th position in the permutation π. Let π'' be the permutation which is obtained by deleting the element n in the permutation π. Obviously if π avoids T then π'' avoids T. Conversely, assume that π'' avoids T but π contains T. So there is an occurrence $inab$ of 1423 should be in π. $i\,(n-1)\,ab$ in π'' is the pattern 1423, a contradiction. We have that π avoids T if and only if π'' avoids T. Hence $a(n;ij) = a(n-1;ij)$. Thus,

$$\begin{aligned}
a(n;i) &= \sum_{j=1}^{i-1} a(n-1;j) + \sum_{j=i+1}^{n-1} a(n-1;ij) + b(n;i) \\
&= \sum_{j=1}^{i-1} a(n-1;j) + a(n-1;i) - \sum_{j=1}^{i-1} a(n-1;ij) + b(n;i) \\
&= \sum_{j=1}^{i} a(n-1;j) - \sum_{j=1}^{i-1} a(n-2;j) + b(n;i).
\end{aligned}$$

Now let $\pi = in\pi' \in S_n(T)$ with $1 \le i \le n-3$. Since π avoids 1423, we can write π as $\pi = inj\pi''$ with either $j = n-1$ or $j \le i-1$. Note that $in(n-1)\pi''$ belongs to $S_n(T)$ if and only if $i(n-1)\pi''$ belongs to $S_{n-1}(T)$. Moreover, if $j \le i-1$ then $\pi = inj\pi'' \in S_n(T)$ if and only if $j(n-1)\pi'''$ avoids T, where π''' is obtained by decreasing the element greater then i by 1. Hence,

$$b(n;i) = \sum_{j=1}^{i} b(n-1;j),$$

with $b(n;n-1)=b(n;n-2)=a(n-2)$ and $b\,(n;n)=0$, which completes the proof. □

Theorem 22. *Let $T=\{1243,1324,1423,2314\}$. The generating function for T-avoiders is given by*

$$F_T(t)=\frac{C(t)}{1-t^3C^5(t)}.$$

Proof. Define $A_n(v)=\sum_{i=1}^n a(n;i)v^{i-1}$ and $B_n(v)=\sum_{i=1}^{n-1} b(n;i)v^{i-1}$. Lemma 21 gives

$$A_n(v)=\frac{1}{1-v}(A_{n-1}(v)-v^nA_{n-1}(1))-\frac{1}{1-v}(vA_{n-2}(v)-v^{n-1}A_{n-2}(1))+B_n(v),$$
$$B_n(v)=\frac{1}{1-v}(B_{n-1}(v)-v^{n-3}B_{n-1}(1))+(v^{n-2}+v^{n-3})A_{n-2}(1)$$

with $A_0(v)=A_1(v)=B_2(v)=1$, $B_0(v)=B_1(v)=0$ and $A_2(v)=1+v$.

Define $A(t;v)=\sum_{n\geq0}A_n(v)t^n$ and $B(t;v)=\sum_{n\geq0}B_n(v)t^n$. The above recurrence relation can be written as

$$A(t;v)=1-t^2+\frac{t}{1-v}(A(t;v)-vA(vt;1))-\frac{vt^2}{1-v}(A(t;v)-A(vt;1))+B(t;v),$$
$$B(t;v)=\frac{t}{1-v}(B(t;v)-B(vt;1)/v^2)+t^2A(vt;1)+t^2(A(vt;1)-1)/v,$$

which is equivalent to

$$\left(1-\frac{t(1-t)}{v(1-v)}\right)A(t/v;v)-B(t/v;v)=1-\frac{t^2}{v^2}-\frac{t(v-t)}{v(1-v)}A(t;1),\tag{1}$$
$$\left(1-\frac{t}{v(1-v)}\right)B(t/v;v)=-\frac{t}{v^3(1-v)}B(t;1)+\frac{t^2(v+1)}{v^3}A(t;1)-\frac{t^2}{v^3}.\tag{2}$$

To solve (2), we apply the kernel method and take $v=\frac{1+\sqrt{1-4t}}{2}=1/C(t)$. Then

$$B(t;1)=t^2(1+C(t))A(t;1)-t^2C(t).\tag{3}$$

By multiplying (1) by $1-\frac{t}{v(1-v)}$ and then using (2) and (3), we obtain

$$\begin{aligned}&\left(1-\frac{t}{v(1-v)}\right)\left(1-\frac{t(1-t)}{v(1-v)}\right)A(t/v;v)\\&\quad=-\frac{t}{v^3(1-v)}\left(t^2(1+C(t))A(t;1)-t^2C(t)\right)+\frac{t^2(v+1)}{v^3}A(t;1)-\frac{t^2}{v^3}\\&\quad+\left(1-\frac{t}{v(1-v)}\right)\left(1-\frac{t^2}{v^2}-\frac{t(v-t)}{v(1-v)}A(t;1)\right).\end{aligned}$$

To solve the preceding functional equation, we apply the kernel method and take $v=1-t$. Then

$$(1-tC^2(t)+t^2C^2(t))A(t,1)=(1-tC(t))C(t),$$

which, by $C(t)=1+tC^2(t)$, completes the proof. □

2.15. **Case 16: $T = \{1243, 1324, 1342, 2314\}$.**

Lemma 23. *For all $1 \le i \le n-2$,*

$$a(n;i) = a(n-1;1) + a(n-1;2) + \cdots + a(n-1;i) + a(i-1)$$

with $a(n;n) = a(n;n-1) = a(n-1)$.

Proof. The initial conditions $a(n;n) = a(n;n-1) = a(n-1)$ easily follow from the definitions. For $1 \le i \le n-2$, we have

$$a(n;i) = \sum_{j=1}^{i-1} a(n;ij) + a(n;i(i+1)) + \sum_{j=i+2}^{n-1} a(n;ij) + a(n;in).$$

So assume $1 \le j < i \le n-2$ and let $\pi = ij\pi'$ be a member of $S_n(T)$. If π avoids T then $j\pi'$ avoids T. Let π contains T and $j\pi'$ avoids T, so there is an occurrence $iabc$ of 2314 in π such that $b < i < a < c$. If $c < j$ then $j\pi'$ contains 2314, otherwise $b > j$ and $j\pi'$ contains 1324, a contradiction. Thus, π avoids T if and only if $j\pi'$ avoids T, which implies $a(n;ij) = a(n-1;j)$.

Let $\pi = i(i+1)\pi'$ be a member of $S_n(T)$, since π avoids 1243, we have that π contains the subsequence $i(i+1)(i+2)\cdots n$. Since π avoids 2314, we can write π as $\pi = i(i+1)\cdots n\pi''$. Therefore, $a(n;i(i+1)) = a(i-1)$.

So assume $1 \le i < j \le n-1$ and let $\pi = ij\pi'$ be a member of $S_n(T)$, then π contains $ij(i+1)n$ or $ijn(i+1)$, which leads to $a(n;ij) = 0$. Moreover, the case $a(n;in) = a(n-1;i)$ easily follow from the definitions (by removing the letter n). Hence, $a(n;i) = a(n-1;1) + a(n-1;2) + \cdots + a(n-1;i) + a(i-1)$, as claimed. □

By Lemmas 17 and 23 with using Theorem 18, we obtain the following result.

Theorem 24. *Let $T = \{1243, 1324, 1342, 2314\}$. The generating function for T-avoiders is given by*

$$F_T(t) = \frac{C(t)}{1 - t^3 C^5(t)}.$$

2.16. **Case 17: $T = \{1234, 1324, 1342, 2314\}$**

Lemma 25. *For all $1 \le i \le n-2$,*

$$a(n;i) = a(n-1;1) + a(n-1;2) + \cdots + a(n-1;i) + a(i-1)$$

with initials $a(n;n) = a(n;n-1) = a(n-1)$.

Proof. The initials could be easily seen. For $1 \le i \le n-2$, we have

$$a(n;i) = \sum_{j=1}^{i-1} a(n;ij) + a(n;i(i+1)) + \sum_{j=i+2}^{n-1} a(n;ij) + a(n;in).$$

The case $a(n;in) = a(n-1;i)$ easily follow from the definitions. For $i+1 < j \le n-1$, the permutation $\pi = ij\pi'$ includes either $ij\,(i+1)\,n$ or $ijn\,(i+1)$ as a subsequence. Hence $a(n;ij) = 0$.

Let $\pi = i\,(i+1)\,\pi'$ be a permutation avoids the pattern set T. Since $1234 \in T$, π contains the subsequence $i\,(i+1)\,n\,(n-1)\cdots(i+2)$. Let $j < i$. Since π avoids the pattern 2314, cannot be located before the element greater than $i+1$. Thus π must be written $\pi = i\,(i+1)\,n\,(n-1)\cdots(i+2)\,\pi''$, which leads to $a(n;i(i+1)) = a\,(i-1)\,.$

So assume $1 \le j < i \le n-2$ and let $\pi = ij\pi' \in S_n\,(T)$. It is clear that $j\pi' \in S_{n-1}\,(T)\,.$ Now assume that $j\pi' \in S_{n-1}\,(T)$ but $\pi = ij\pi' \notin S_n\,(T)$. So π contains the pattern 2314. Let $iabc$ be this pattern. We have $b < i < a < c$. If $j > b$ then $jabc$ is the pattern 2314, otherwise $j < b$ then $j\pi'$ contains 1324. There is a contradiciton with $j\pi' \in S_{n-1}\,(T)$. Finally we obtain $\pi = ij\pi' \in S_n\,(T)$ if and only if $j\pi' \in S_{n-1}\,(T)\,,$ which gives $a(n;ij) = a(n-1;j)$. So the proof is completed. □

By Lemmas 17 and 25 with using Theorem 18, we obtain the following result.

Theorem 26. *Let $T = \{1234, 1324, 1342, 2314\}$. The generating function for T-avoiders is given by*

$$F_T(t) = \frac{C(t)}{1 - t^3C^5(t)}.$$

2.17. Case 18: $T = \{1342, 2341, 3241, 3421\}$.

Theorem 27. *Let $T = \{1342, 2341, 3241, 3421\}$. The generating function for T-avoiders is given by*

$$F_T(t) = \frac{C(t)}{1 - t^3C^5(t)}.$$

Proof. Let $G_m(t)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(t) = 1$ and $G_1(t) = tF_T(t)$.

Let us write an equation for $G_2(x)$. Let $\pi = i\pi'n\pi'' \in S_n(T)$ with exactly 2 left-right maxima. Since π avoids 3421, we see that all the letters that are smaller than i belongs to π'' are increasing. Since π avoids 3241, we obtain that π contains the subsequence $(i-d)(i-d+1)\cdots(i-1)$ and $\pi' < i-d$. Since π avoids 2341 an 1342, we see that π'' can be written as

$$\begin{aligned}\pi'' = p_1p_2\cdots p_{s_1}(i-d)p_{s_1+1}p_{s_1+2}\cdots p_{s_2}(i-d+1)\\ \cdots p_{s_d+1}p_{s_d+2}\cdots p_{s_{d+1}}(i-1)\alpha^{(s_{d+1}+1)}\alpha^{(s_{d+1})}\cdots\alpha^{(1)}\end{aligned}$$

such that $p_{j-1} > p_j$ and $\alpha^{(j+1)} < p_j < \alpha^{(j)}$ for all $j = 1, 2, \ldots, s_{d+1}$. Note that π avoids T if and only if $\alpha^{(j)}$ avoids 231 for all j and π' avoids T. Therefore, we have a contribution of

$$\frac{t^{2+d}C(t)F_T(t)}{(1 - tC(t))^d} = t^{2+d}C^{d+1}(t)F_T(t),$$

for fixed d. Hence,

$$G_2(t) = \sum_{d \ge 0} t^{2+d}C^{d+1}(t)F_T(t) = t^2C^2(t)F_T(t).$$

Let us write an equation for $G_m(t)$ for all $m \ge 3$. Let $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima such that $i_1 < i_2 < \cdots < i_m = n$. Since π avoids 2341 and 1342, we see that $\pi^{(j)} > i_{j-1}$ for all $j = 2, 3, \ldots, m$. So, π avoids T if and only if $i_1\pi^{(1)}i_2\pi^{(2)}$ avoids T, and $\pi^{(j)}$ avoids 231 for all $j = 3, 4, \ldots, m$. Hence,

$$G_m(t) = t^{m-2}G_2(t)C^{m-2}(t).$$

By summing over $m\geq 2$, we obtain

$$F_T(t)-1-tF_T(t)=\sum_{m\geq 2}t^{m-2}G_2(t)C^{m-2}(t)=G_2(t)C(t)=t^2C^3(t)F_T(t),$$

which, by using the fact that $C(t)=1+tC^2(t)$, leads to $F_T(t)=\frac{C(t)}{1-t^3C^5(t)}$, as claimed. □

2.18. Case 19: $T=\{1243,1342,1423,2341\}$.

Lemma 28. *Let $b(n;i)=a(n;in)$. For all $1\leq i\leq n-2$,*

$$a(n;i)=\sum_{j=1}^{i}a(n-1;j)-\sum_{j=1}^{i-1}a(n-2;j)+b(n;i)$$

with $a(n;n)=a(n;n-1)=a(n-1)$, where for all $1\leq i\leq n-3$,

$$b(n;i)=b(n-1;1)+\cdots+b(n-1;i)$$

with $b(n;n-1)=b(n;n-2)=a(n-2)$.

Proof. The initial conditions $a(n;n)=a(n;n-1)=a(n-1)$, $b(n;n-1)=a(n;(n-1)n)=a(n-2)$ and $b(n;n-2)=a(n;(n-2)n)=a(n-2)$ easily follow from the definitions. For $1\leq i\leq n-2$, we have

$$a(n;i)=\sum_{j=1}^{i-1}a(n;ij)+\sum_{j=i+1}^{n-1}a(n;ij)+a(n;in).$$

So assume $1\leq j<i\leq n$ and let $\pi=ij\pi'$ be a member of $S_n(T)$. If π avoids T then $j\pi'$ avoids T. Let π contains T and $j\pi'$ avoids T, so there is an occurrence $iabc$ of 2341 in π such that $c<i<a<b$. If $c<j$ then $j\pi'$ contains 2341, otherwise $c>j$ and $j\pi'$ contains 1342, a contradiction. Thus, π avoids T if and only if $j\pi'$ avoids T, which implies $a(n;ij)=a(n-1;j)$. Let $\pi=ij\pi'$ be a member of $S_n(T)$ such that $1\leq i<j\leq n-1$, since π avoids 1243, we have that π contains the subsequence $ij(j+1)\cdots n$. Since π avoids 2341 and $j\leq n-1$, we have that n is the rightmost letter of π, so, $a(n;ij)=a(n-1;ij)$. Thus,

$$\begin{aligned}a(n;i)&=\sum_{j=1}^{i-1}a(n-1;j)+\sum_{j=i+1}^{n-1}a(n-1;ij)+b(n;i)\\&=\sum_{j=1}^{i-1}a(n-1;j)+a(n-1;i)-\sum_{j=1}^{i-1}a(n-1;ij)+b(n;i)\\&=\sum_{j=1}^{i}a(n-1;j)-\sum_{j=1}^{i-1}a(n-2;j)+b(n;i).\end{aligned}$$

Now let $\pi=in\pi'\in S_n(T)$ with $1\leq i\leq n-3$. Since π avoids 1423, we can write π as $\pi=inj\pi''$ with either $j=n-1$ or $j\leq i-1$. Note that $in(n-1)\pi''$ belongs to $S_n(T)$ if and only if $i(n-1)\pi''$ belongs to $S_{n-1}(T)$. Moreover, if $j\leq i-1$ then $\pi=inj\pi''\in S_n(T)$ if and only if $j(n-1)\pi''$ avoids T (π contains the subsequence $j(n-1)(n-2)\cdots(i+1)(i-1)\cdots(j+1)$). Hence, $b(n;i)=\sum_{j=1}^{i}b(n-1;j)$ with $b(n;n-1)=b(n;n-2)=a(n-2)$, which completes the proof. □

By Lemmas 21 and 28 with using Theorem 22, we obtain the following result.

Theorem 29. *Let* $T = \{1243, 1342, 1423, 2341\}$. *The generating function for* T*-avoiders is given by*

$$F_T(t) = \frac{C(t)}{1 - t^3 C^5(t)}.$$

Moreover, we have the following result for all pattern sets A_j, where $j = 1, 2, ..., 19$.

Theorem 30. *The number of* A_j*-avoiders of size* n*, where* $j = 1, 2, ..., 19$. *is given by*

$$a(n) = \sum_{\ell=0}^{n/3} \frac{5\ell + 1}{n + 2\ell + 1} \binom{2n - \ell}{n + 2\ell}.$$

Proof. By [8, Eq. 2.5.16], we can see that

$$\frac{C(t)}{1 - t^3 C^5(t)} = \sum_{\ell \geq 0} t^{3\ell} C^{\ell+1}(t) = \sum_{\ell \geq 0} \sum_{j \geq 0} \frac{(5\ell + 1)(2j + 5\ell)!}{j!(j + 5\ell + 1)!} t^{j+3\ell},$$

which completes the proof. □

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