Evaluation of spectrum of 2-periodic tridiagonal-Sylvester matrix

Emrah KILIÇ AND Talha ARIKAN*

ABSTRACT. The Sylvester matrix was firstly defined by J.J. Sylvester. Some authors have studied the relationships between certain orthogonal polynomials and determinant of Sylvester matrix. Chu studied a generalization of the Sylvester matrix. In this paper, we introduce its 2-period generalization. Then we compute its spectrum by left eigenvectors with a similarity trick.

1. INTRODUCTION

There has been increasing interest in tridiagonal matrices in many different theoretical fields, especially in applicative fields such as numerical analysis, orthogonal polynomials, engineering, telecommunication system analysis, system identification, signal processing (e.g., speech decoding, deconvolution), special functions, partial differential equations and naturally linear algebra (see [2, 4, 5, 6, 15]). Some authors consider a general tridiagonal matrix of finite order and then describe its LU factorizations, determine the determinant and inverse of a tridiagonal matrix under certain conditions (see [3, 7, 10, 11]).

The Sylvester type tridiagonal matrix $M_n(x)$ of order $(n+1)$ is defined as

$$M_n(x) = \begin{bmatrix} x & 1 & 0 & 0 & \cdots & 0 & 0 \\ n & x & 2 & 0 & \cdots & 0 & 0 \\ 0 & n-1 & x & 3 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & n-1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & x & n \\ 0 & 0 & 0 & 0 & \cdots & 1 & x \end{bmatrix}$$

1991 *Mathematics Subject Classification.* 15A36, 15A18, 15A15.
Key words and phrases. Sylvester Matrix, spectrum, determinant.
* Corresponding author.

and Sylvester [12] gave its determinant as

$$\det M_n(x) = \prod_{k=0}^{n} (x + n - 2k).$$

Askey [1] showed two ways to compute the determinant of $M_n(x)$, one matrix-theoretic and another based on orthogonal polynomials. Also he explored their connection to orthogonal polynomials. For the relationships between orthogonal polynomials and other determinants of Sylvester type matrices related to Krawtchouk, Hahn and Racah polynomials, we refer to [14]. Holtz [8] showed how the determinants in [12] can be evaluated by left eigenvectors of corresponding matrices coupled with a simple similarity trick.

Chu [13] generalized the Sylvester matrix by adding a new parameter as shown

$$M_n(x,y) = \begin{bmatrix} x & 1 & & & & 0 \\ n & x+y & 2 & & & \\ & n-1 & x+2y & \ddots & & \\ & & \ddots & \ddots & n-1 & \\ & & & 2 & x+(n-1)y & n \\ 0 & & & & 1 & x+ny \end{bmatrix}$$

and by using the method which Holtz uses in [8], evaluated its determinant as

$$\det M_n(x,y) = \prod_{k=0}^{n} \left(x + \frac{ny}{2} + \frac{n-2k}{2}\sqrt{4+y^2}\right),$$

via the generalized Fibonacci sequences.

In this paper, we consider a new generalization of tridiogonal-Sylvester matrix. Then we compute its spectra and also determinant.

2. A Periodic Tridiagonal-Sylvester Matrix

We define 2-period Sylvester matrix of order $(n+1)$ as follows :

$$A_n(x,y) = \begin{bmatrix} x & 1 & & & & 0 \\ n & y & 2 & & & \\ & n-1 & x & \ddots & & \\ & & \ddots & \ddots & n-1 & \\ & & & 2 & a_{n-1}(x,y) & n \\ 0 & & & & 1 & a_n(x,y) \end{bmatrix},$$

where

$$a_n(x,y) = \begin{cases} x & \text{if } n \text{ is even,} \\ y & \text{if } n \text{ is odd.} \end{cases}$$

If we take $x = y$, then the matrix $A_n(x,x)$ gives the Sylvester matrix $M_n(x)$. Kılıç [9] studied the case $y = -x$.

In this paper, our main purpose is to prove determinant formula for the matrix $A_n(x,y)$:

$$\det A_n(x,y) = \begin{cases} x \prod_{t=1}^{n/2} (xy - 4t^2) & \text{if } n \text{ is even,} \\ \prod_{t=0}^{\lfloor n/2 \rfloor} (xy - (2t+1)^2) & \text{if } n \text{ is odd.} \end{cases}$$

We will frequently denote the matrix $A_n(x,y)$ by A_n and, $a_n(x,y)$ by a_n.

Let $\lambda_1 = \frac{1}{2}(x+y) + \frac{1}{2}\delta$ and $\lambda_2 = \frac{1}{2}(x+y) - \frac{1}{2}\delta$ where $\delta = \sqrt{(x-y)^2 + (2n)^2}$.

For the matrix A_n of order $n+1$ with odd n, define the vectors with $(n+1)$ dimension:

$$z^+ := \begin{bmatrix} 1 & \frac{(y-x)+\delta}{2n} & 1 & \frac{(y-x)+\delta}{2n} & \cdots & 1 & \frac{(y-x)+\delta}{2n} \end{bmatrix}$$

and

$$z^- := \begin{bmatrix} 1 & \frac{(y-x)-\delta}{2n} & 1 & \frac{(y-x)-\delta}{2n} & \cdots & 1 & \frac{(y-x)-\delta}{2n} \end{bmatrix}.$$

For the matrix A_n of order $n+1$ with even n, define the vectors with $n+1$ dimension:

$$s^+ := \begin{bmatrix} 1 & \frac{(y-x)+\delta}{2n} & 1 & \frac{(y-x)+\delta}{2n} & \cdots & 1 & \frac{(y-x)+\delta}{2n} & 1 \end{bmatrix}$$

and

$$s^- := \begin{bmatrix} 1 & \frac{(y-x)-\delta}{2n} & 1 & \frac{(y-x)-\delta}{2n} & \cdots & 1 & \frac{(y-x)-\delta}{2n} & 1 \end{bmatrix}.$$

We need the following results:

Lemma 1. *For odd $n > 0$, the matrix A_n has the eigenvalues λ_1 and λ_2 with the corresponding left eigenvectors z^+ and z^-, respectively.*

Proof. To prove the claim, it is sufficient to show $z^+ A_n = \lambda_1 z^+$ and $z^- A_n = \lambda_2 z^-$. From the definition of A_n, we should prove that the k^{th} components of $z^{\pm} A_n$ are

$$\begin{aligned} z_0^{\pm} x + z_1^{\pm} n &= z_0^{\pm} \lambda_{1,2} \quad \text{for } k = 0, \\ z_{n-1}^{\pm} n + z_n^{\pm} y &= z_n^{\pm} \lambda_{1,2} \quad \text{for } k = n, \end{aligned}$$

and for $0 < k < n$,

$$k z_{k-1}^{\pm} + a_k z_k^{\pm} + (n-k) z_{k+1}^{\pm} = z_k^{\pm} \lambda_{1,2},$$

where a_n is defined as before.

For the case $k=0$, we get

$$x+n\frac{1}{2n}[(y-x)\pm\delta]=\frac{1}{2}(x+y)\pm\frac{1}{2}\delta=\lambda_{1,2},$$

as claimed. Now we consider the case $k=n$ and examine the equality $z^+A_n=\lambda_1 z^+$. So we get

$$z_n^+\lambda_1=\left(\frac{(y-x)+\delta}{2n}\right)\left(\frac{(x+y)+\delta}{2}\right) \tag{2.1}$$

$$=\frac{1}{2n}(y^2-xy+y\delta+2n^2)$$

$$=n+\frac{y}{2n}\left((y-x)+\delta\right) \tag{2.2}$$

$$=z_{n-1}^+n+z_n^+y,$$

as claimed. To complete the proof, we show the last case $0<k<n$. Now we examine this case under two conditions: For even k,

$$\begin{aligned}&kz_{k-1}^+ + xz_k^+ + (n-k)z_{k+1}^+\\&=k\left(\frac{1}{2n}((y-x)+\delta)\right)+x+(n-k)\left(\frac{1}{2n}((y-x)+\delta)\right)\\&=x+n\left(\frac{1}{2n}((y-x)+\delta)\right)\\&=\frac{1}{2}(x+y)+\frac{1}{2}\delta=\lambda_1=z_k^+\lambda_1,\end{aligned}$$

and for odd k,

$$\begin{aligned}kz_{k-1}^{\pm}+yz_k^{\pm}+(n-k)z_{k+1}^{\pm}&=k+y\left(\frac{1}{2n}((y-x)+\delta)\right)+(n-k)\\&=n+y\left(\frac{1}{2n}((y-x)+\delta)\right),\end{aligned}$$

which, by the equations (2.1) and (2.2), equals

$$\left(\frac{1}{2n}\left((y-x)+\delta\right)\right)\left(\frac{1}{2}\left((x+y)+\delta\right)\right)=z_k^+\lambda_1.$$

So the proof is completed for the case $z^+A_n=\lambda_1 z^+$. The other case, $z^-A_n=\lambda_2 z^-$, can be similarly shown. □

Lemma 2. *For even $n>0$, the matrix A_n has the eigenvalues λ_1 and λ_2 with the corresponding to the left eigenvectors s^+ and s^-, respectively.*

Proof. The proof can be done similar to the proof of the previous Lemma. □

For odd $n > 0$, we define a $(n+1)\times(n+1)$ matrix T_n as

$$T_n = \begin{bmatrix} z_0^+ & z_1^+ & \vdots & z_2^+ & \cdots & z_{n-1}^+ & z_n^+ \\ z_0^- & z_1^- & \vdots & z_2^- & \cdots & z_{n-1}^- & z_n^- \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & \vdots & & & & \\ 0_{(n-1)\times 2} & & \vdots & & I_{n-1} & & \end{bmatrix},$$

where $0_{(n-1)\times 2}$ is the zero matrix of order $(n-1)\times 2$ and I_n is the identity matrix of order n.

We can obtain inverse of the matrix T_n as follows:

$$T_n^{-1} = \begin{bmatrix} \frac{(x-y)+\delta}{2\delta} & \frac{\delta-(x-y)}{2\delta} & \vdots & -1 & 0 & -1 & 0 & \cdots & -1 & 0 \\ \frac{n}{\delta} & -\frac{n}{\delta} & \vdots & 0 & -1 & 0 & -1 & \cdots & 0 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & 0_{(n-1)\times 2} & \vdots & & & & & & & \\ & & \vdots & & I_{n-1} & & & & & \\ & & \vdots & & & & & & & \end{bmatrix}.$$

So we can see that the matrix A_n is similar to the matrix $E_n := T_n A_n T_n^{-1}$ via the matrix T_n as shown

$$T_n A_n T_n^{-1} = \begin{bmatrix} \lambda_1 & 0 & \vdots & 0_{2\times(n-1)} \\ 0 & \lambda_2 & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ \frac{n(n-1)}{\delta} & -\frac{n(n-1)}{\delta} & \vdots & \\ 0_{(n-2)\times 2} & & \vdots & W_{n-1} \end{bmatrix},$$

where the matrix W_{n-1} of order $(n-1)$ is given by

$$W_{n-1} = \begin{bmatrix} x & 4-n & 0 & 1-n & \cdots & 0 & 1-n \\ n-2 & y & 4 & 0 & \cdots & \cdots & 0 \\ 0 & n-3 & x & 5 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 3 & y & n-1 & 0 \\ \vdots & & & \ddots & 2 & x & n \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & y \end{bmatrix}.$$

Considering the 2×2 principal submatrix of E_n, it is clearly seen that λ_1 and λ_2 are two eigenvalues of E_n.

We focus on the matrix A_n for odd n up to now. By consider the matrix A_n for even n, we define a matrix Y_n of order $(n+1)$ as shown:

$$Y_n = \begin{bmatrix} s_0^+ & s_1^+ & \vdots & s_2^+ & \cdots & s_{n-1}^+ & s_n^+ \\ s_0^- & s_1^- & \vdots & s_2^- & \cdots & s_{n-1}^- & s_n^- \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \\ 0_{(n-1)\times 2} & & \vdots & & I_{n-1} & & \end{bmatrix},$$

where $0_{(n-1)\times 2}$ and I_n are defined as before.

We also obtain the inverse matrix Y_n^{-1} in the form

$$Y_n^{-1} = \begin{bmatrix} \frac{x-y+\delta}{2\delta} & -\frac{x-y-\delta}{2\delta} & \vdots & -1 & 0 & -1 & \cdots & 0 & -1 \\ \frac{n}{\delta} & -\frac{n}{\delta} & \vdots & 0 & -1 & 0 & \cdots & -1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{(n-1)\times 2} & & \vdots & & I_{n-1} & & & & \end{bmatrix}.$$

So the matrix A_n is similar to matrix $D_n := Y_n A_n Y_n^{-1}$ via the matrix Y_n is given by

$$Y_n A_n Y_n^{-1} = \begin{bmatrix} \lambda_1 & 0 & \vdots & 0_{2\times(n-1)} \\ 0 & \lambda_2 & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ \frac{n(n-1)}{\delta} & -\frac{n(n-1)}{\delta} & \vdots & \\ 0_{(n-2)\times 2} & & \vdots & Q_{n-1} \end{bmatrix},$$

where the matrix Q_{n-1} of order $(n-1)$ is given by

$$Q_{n-1} = \begin{bmatrix} x & 4-n & 0 & 1-n & \cdots & 0 & 1-n & 0 \\ n-2 & y & 4 & 0 & \cdots & & \cdots & 0 \\ 0 & n-3 & x & 5 & \ddots & & & \vdots \\ \vdots & 0 & n-4 & y & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & n-2 & \ddots & \vdots \\ & & & \ddots & 3 & x & n-1 & 0 \\ \vdots & & & & \ddots & 2 & y & n \\ 0 & \cdots & & & \cdots & 0 & 1 & x \end{bmatrix}.$$

Consequently, by the above results, the matrix A_n has two eigenvalues λ_1 and λ_2 for each n. To compute remaining eigenvalues of matrix A_n, we will give some auxiliary results.

Now we define an upper triangular matrix U_n of order n as follows

$$U_n = \begin{bmatrix} 1 & 0 & -1 & 0 & \cdots & 0 \\ & 1 & 0 & -1 & \ddots & \vdots \\ & & 1 & 0 & \ddots & 0 \\ & & & \ddots & \ddots & -1 \\ & & & & 1 & 0 \\ 0 & & & & & 1 \end{bmatrix}.$$

and U_n^{-1} can be found as follows for even n,

$$U_n^{-1} = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ & 1 & 0 & 1 & \ddots & & 1 \\ & & 1 & 0 & & & 0 \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & \ddots & \ddots & 1 \\ & & & & & \ddots & 0 \\ 0 & & & & & & 1 \end{bmatrix}$$

and for odd n, the matrix U_n^{-1} takes the form:

$$U_n^{-1} = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 0 & 1 \\ & 1 & 0 & 1 & \ddots & \ddots & 0 \\ & & 1 & 0 & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & 1 & 0 \\ & & & & 1 & 0 & 1 \\ & & & & & 1 & 0 \\ 0 & & & & & & 1 \end{bmatrix}.$$

Then both the matrix W_n and Q_n are similar to the same tridiagonal matrix G_n of order n, that is, they satisfied the equations

$$G_n := U_n^{-1} W_n U_n \quad \text{and} \quad G_n := U_n^{-1} Q_n U_n,$$

with

$$G_n = \begin{bmatrix} x & 1 & & & & & 0 \\ n-1 & y & 2 & & & & \\ & n-2 & x & 3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 3 & a_{n-3} & n-2 & \\ & & & & 2 & a_{n-2} & n-1 \\ 0 & & & & & 1 & a_{n-1} \end{bmatrix}$$

and a_n is defined as before.

For further computations, we define a $(n+1)\times(n+1)$ matrix U via the matrix U_n as follows

$$U=\begin{bmatrix} I_2 & \vdots & 0_{2\times(n-1)} \\ \cdots & \vdots & \cdots \\ 0_{(n-1)\times 2} & \vdots & U_{n-1} \end{bmatrix},$$

and then it can be easily seen

$$U^{-1}=\begin{bmatrix} I_2 & \vdots & 0_{2\times(n-1)} \\ \cdots & \vdots & \cdots \\ 0_{(n-1)\times 2} & \vdots & U_{n-1}^{-1} \end{bmatrix}.$$

For both even and odd cases, we get

$$U^{-1}E_nU=\begin{bmatrix} \lambda_1 & 0 & \vdots & 0_{2\times(n-1)} \\ 0 & \lambda_2 & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ \frac{n(n-1)}{\delta} & -\frac{n(n-1)}{\delta} & \vdots & \\ 0_{(n-2)\times 2} & & \vdots & U_{n-1}^{-1}W_{n-1}U_{n-1} \end{bmatrix}$$

and

$$U^{-1}D_nU=\begin{bmatrix} \lambda_1 & 0 & \vdots & 0_{2\times(n-1)} \\ 0 & \lambda_2 & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ \frac{n(n-1)}{\delta} & -\frac{n(n-1)}{\delta} & \vdots & \\ 0_{(n-2)\times 2} & & \vdots & U_{n-1}^{-1}Q_{n-1}U_{n-1} \end{bmatrix}.$$

In general, we obtain that $U^{-1}E_nU$ and $U^{-1}D_nU$ are reduced to a block form:

$$U^{-1}E_nU=U^{-1}D_nU=\begin{bmatrix} \lambda_1 & 0 & \vdots & 0_{2\times(n-1)} \\ 0 & \lambda_2 & \vdots & \\ \cdots & \cdots & \cdots & \cdots \\ \frac{n(n-1)}{\delta} & -\frac{n(n-1)}{\delta} & \vdots & \\ 0_{(n-2)\times 2} & & \vdots & G_{n-1} \end{bmatrix}, \tag{2.3}$$

where G_n is defined as before.

Up to now, the following results have been obtained

$$\begin{aligned} E_n &= T_n A_n T_n^{-1} \quad \text{for odd } n, \\ D_n &= Y_n A_n Y_n^{-1} \quad \text{for even } n, \\ G_n &= U_n^{-1} W_n U_n \quad \text{for odd } n, \\ G_n &= U_n^{-1} Q_n U_n \quad \text{for even } n. \end{aligned} \tag{2.4}$$

From the definition of G_n, one can see that $G_n = A_{n-1}$ and both of $U^{-1}E_nU$ and $U^{-1}D_nU$ can be rewritten in the following lower-triangular form

$$\begin{bmatrix} Diag(\lambda_1, \lambda_2) & 0 \\ * & A_{n-2} \end{bmatrix}.$$

From (2.3) and (2.4) we get the following recurrence relation for $\det A_n$

$$\begin{aligned} \det A_0 &= x, \\ \det A_1 &= xy - 1, \\ \det A_n &= \lambda_1\lambda_2 \det A_{n-2} = (xy - n^2) \det A_{n-2}, \end{aligned}$$

for $n > 0$.

Therefore we have the spectrum of the matrix A_n : for even n,

$$\lambda(A_n) = \left\{ \frac{1}{2}(x+y) \mp \frac{1}{2}\sqrt{(x-y)^2 + (4k)^2} \right\}_{k=1}^{n/2} \cup \{x\}$$

and for odd n,

$$\lambda(A_n) = \left\{ \frac{1}{2}(x+y) \mp \frac{1}{2}\sqrt{(x-y)^2 + (4k+2)^2} \right\}_{k=0}^{\lfloor n/2 \rfloor}.$$

By considering spectrum of the matrix A_n and recurrence relation of $\det A_n$, we deduce that for even n,

$$\det A_n(x,y) = x\prod_{t=1}^{n/2}(xy - (2t)^2)$$

and for odd n,

$$\det A_n(x,y) = \prod_{t=0}^{\lfloor n/2 \rfloor}(xy - (2t+1)^2).$$

As we stated earlier, if we take $x = y$, then for even n,

$$\det A_n(x,x) = x\prod_{t=1}^{n/2}(x^2 - (2t)^2)$$

and for odd n,

$$\det A_n(x,x) = \prod_{t=0}^{\lfloor n/2 \rfloor} (x^2 - (2t+1)^2),$$

which, by combining, give us the single formula

$$\det M_n(x) = \prod_{k=0}^{n} (x+n-2k),$$

which equals to $\det M_n(x)$.

Note that if we take $y = -x$ then we obtain results in [9].

References

[1] Askey R. Evaluation of Sylvester type determinants using orthogonal polynomials. Advances in analysis. Hackensack, NJ : World Scientific 2005: 116.

[2] Bunse-Gerstner A, Byers R, Mehrmann V. A chart of numerical methods for structured eigenvalue problems. SIAM J Matrix Anal Appl 1992; 13 (2): 419–453.

[3] Burden RL, Fairs JD. Reynolds AC. Numerical Analysis, Second ed., Prindle, Weber & Schmidt, Boston: MA, 1981.

[4] El-Mikkawy M. A fast algorithm for evaluating nth order tri-diagonal determinants. J Comput Appl Math 2004; 166(2): 581–584.

[5] El-Mikkawy M, Karawia A. Inversion of general tridiagonal matrices. Appl Math Letters 2006; 19(8): 712–720.

[6] Fischer CF, Usmani RA. Properties of some tridiagonal matrices and their application to boundary value problems. SIAM J Numer Anal 1969; 6 (1): 127–142.

[7] Hager WW. Applied Numerical Linear Algebra, Prentice-Hall International Editions, Englewood Cliffs: NJ, 1988.

[8] Holtz O. Evaluation of Sylvester type determinants using block-triangularization. Advances in analysis. Hackensack , NJ : World Scientific 2005: 395–405.

[9] Kılıç E. Sylvester-tridiagonal matrix with alternating main diagonal entries and its spectra. Inter J Nonlinear Sci Num Simulation 2013; 14(5): 261–266.

[10] Lehmer DH. Fibonacci and related sequences in periodic tridiagonal matrices. The Fibonacci Quart 1975; 13: 150–158.

[11] Meurant G. A review on the inverse of symmetric tridiagonal and block tridiagonal matrices. SIAM J Matrix Anal Appl 1992; 13(3): 707–728.

[12] Sylvester JJ. Theoreme sur les determinants. Nouvelles Annales de Math 1854; 13: 305.

[13] Wenchang C. Fibonacci polynomials and Sylvester determinant of tridiagonal matrix. Appl Math Comput 2010; 216(3): 1018–1023.

[14] Wenchang C. Eigenvectors of tridiagonal matrices of Sylvester type. Calcolo 2008; 45(4): 217–233.

[15] Zhao H, Zhu G, Xiao P. A backward three-term recurrence relation for vector valued continued fractions and its applications. J Comput Appl Math 2002; 142(2): 389–400.

Department of Mathematics, TOBB Economics and Technology University 06560, Ankara, Çankaya, Turkey
E-mail address: ekilic@etu.edu.tr

Department of Mathematics, Hacettepe University, Ankara, Çankaya, Turkey
E-mail address: tarikan@hacettepe.edu.tr