# THE GENERALIZED $q$-PILBERT MATRIX

EMRAH KILIÇ AND HELMUT PRODINGER

ABSTRACT. A generalized $q$-Pilbert matrix from [2] is further generalized, introducing one additional parameter. Explicit formulæ are derived for the LU-decomposition and their inverses, as well as the Cholesky decomposition. The approach is to use $q$-analysis and to leave the justification of the necessary identities to the $q$-version of Zeilberger's celebrated algorithm. However, the necessary identities have appeared already in [2] in disguised form, so that no new computations are necessary.

## 1. INTRODUCTION

The Filbert matrix $H_n = \left(\check{h}_{ij}\right)_{i,j=1}^{n}$ is defined by $\check{h}_{ij} = \frac{1}{F_{i+j-1}}$ as an analogue of the Hilbert matrix where $F_n$ is the $n$th Fibonacci number. It has been defined and studied by Richardson [4].

In [1], Kılıç and Prodinger studied the generalized matrix with entries $\frac{1}{F_{i+j+r}}$, where $r \geq -1$ is an integer parameter. They gave its LU factorization and, using this, computed its determinant and inverse. Also the Cholesky factorization was derived. After this generalization, Prodinger [3] defined a new generalization of the generalized Filbert matrix by introducing 3 additional parameters. Again, explicit formulæ for the LU-decomposition, their inverses, and the Cholesky factorization were derived.

Recently, in [2], Kılıç and Prodinger give a further generalization of the generalized Filbert Matrix $\mathcal{F}$ with entries $\frac{1}{F_{i+j+r}}$, where $r \geq -1$ is an integer parameter. They define the matrix $\mathcal{Q}$ with entries $h_{ij}$ as follows

$$h_{ij} = \frac{1}{F_{i+j+r}F_{i+j+r+1}\dots F_{i+j+r+k-1}},$$

where $r \geq -1$ is an integer parameter and $k \geq 0$ is an integer parameter.

When $k = 1$, we get the generalized Filbert Matrix $\mathcal{F}$, as studied before. They derive explicit formulæ for the LU-decomposition and their inverses. Again, explicit formulæ for the LU-decomposition, their inverses, and the Cholesky factorization were derived.

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In this paper, we introduce a new kind generalization of the Filbert matrix $\mathcal{F}$ and define the matrix $\mathcal{G}$ with enties $g_{ij}$ by

$$g_{ij} = \frac{1}{F_{\lambda(i+j)+r}F_{\lambda(i+j+1)+r}\dots F_{\lambda(i+j+k-1)+r}},$$

where $r > -1$ and $\lambda > 1$ are integer parameters.

Here we note that the case $\lambda = 1$ was given in [2] so that we shall study the case $\lambda > 1$ throughout this paper. However, all the old results are covered as well, if in some cases the resulting formula is interpreted as a limit.

Our approach will be as follows. We will use the Binet form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q},$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$.

Throughout this paper we will use the following notations: the $q$-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1})$ and as usual for $z > 1$, the Gaussian $q$-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_{(z,y)} = \frac{(q^z;q^y)_n}{(q^z;q^y)_k(q^z;q^y)_{n-k}}$$

and for the case $z = y$, we will denote the Gaussian $q$-binomial coefficients as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{z} = \frac{(q^z;q^z)_n}{(q^z;q^z)_k(q^z;q^z)_{n-k}}.$$

Here we should note that when $z = 1$, $(q^z;q^y)_n$ would be zero in some cases so that $\begin{bmatrix} n \\ k \end{bmatrix}_{(z,y)}$ would be indefinite. In order to prevent such cases, we will consider the Gaussian $q$-binomial coefficients for $z > 1$. Furthermore, for the matrix $\mathcal{F}$ and its properties with $z = 1$, we can refer [2].

Considering the definitions of the matrix $\mathcal{G}$ and the $q$-Pochhammer symbol, we rewrite the matrix $\mathcal{G} = [g_{ij}]$ for $\lambda \geq 1$ as

$$g_{ij} = \mathbf{i}^{k(\lambda(i+j)+r-1)+\frac{\lambda k(k-1)}{2}} q^{-\frac{k}{2}(\lambda(i+j)+r-1)-\frac{\lambda k(k-1)}{4}} \frac{\left(q^{\lambda(i+j)+r};q^{\lambda}\right)_k}{(1-q)^k}.$$

We call the matrix $\mathcal{G}_n$ the *generalized $q$-Pilbert matrix.* (When $\lambda = 1$, we get the generalized Filbert Matrix $\mathcal{Q}$, as studied before.)

We will derive explicit formulæ for the LU-decomposition and their inverses. Similarly to the results of [1, 2], the size of the matrix does not really matter, and it can be thought about an infinite matrix $\mathcal{G}$ and restrict it whenever necessary to the first $n$ rows resp. columns and write $\mathcal{G}_n$. The entries of the inverse matrix $\mathcal{G}_n^{-1}$ are not closed form expressions, as in our previous paper [1, 2], but can only be given as a (simple) sum. We also provide the Cholesky decomposition.

All the identities we will obtain hold for general $q$, and results about Fibonacci numbers come out as corollaries for the special choice of $q$.

Furthermore, we will use *generalized Fibonomial coefficients*

$$\left\{ {n \atop k} \right\}_{(a,b)} = \frac{F_{b(n-1)+a}F_{b(n-2)+a}\dots F_{b(n-k)+a}}{F_a F_{b+a} F_{2b+a}\dots F_{b(k-1)+a}}$$

with $\left\{ {n \atop 0} \right\}_{(a,b)} = 1$ where $F_n$ is the $n$th Fibonacci number.

For $a = b$, we denote the generalized Fibonomial coefficents as $\left\{ {n \atop k} \right\}_a$. Especially for $a = b = 1$, the generalized Fibonomial coefficients are reduced to the usual Fibonomial coefficients denoted by $\left\{ {n \atop k} \right\}$ :

$$\left\{ {n \atop k} \right\} = \frac{F_n F_{n-1}\dots F_{n-k+1}}{F_1 F_2 \dots F_k}.$$

The link between the generalized Fibonomial and Gaussian $q$-binomial coefficients is

$$\left\{ {n \atop k} \right\}_{(z,y)} = \alpha^{yk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{(z,y)} \quad \text{with} \quad q = -\alpha^{-2}.$$

We will obtain the LU-decomposition $\mathcal{G} = L \cdot U$, where $L = (l_{ij})$ and $U = (u_{ij})$ :

**Theorem 1.** *For $1 \le d \le n$ we have*

$$l_{n,d} = \mathbf{i}^{\lambda k(d-n)} q^{\lambda \frac{k(n-d)}{2}} \frac{(q^\lambda; q^\lambda)_{n-1}(q^{\lambda(d+1)+r}; q^\lambda)_{d+k-1}}{(q^\lambda; q^\lambda)_{d-1}(q^\lambda; q^\lambda)_{n-d}(q^{\lambda(n+1)+r}; q^\lambda)_{d+k-1}}.$$

As a Fibonacci consequence of Theorem 1, we have

**Corollary 1.** *For $1 \le d \le n$,*

$$l_{n,d} = \left\{ {n-1 \atop d-1} \right\}_\lambda \left\{ {2d+k \atop d+1} \right\}_{(r,\lambda)} \left\{ {n+d+k \atop n+1} \right\}_{(r,\lambda)}^{-1}.$$

From the Corollary above, we have the following examples: For $\lambda = 2$, $r = -1$,

$$\begin{aligned} l_{n,d} = &\left\{ {n-1 \atop d-1} \right\}_2 \left\{ {n+d+k-2 \atop d+k-1} \right\}_2 \left\{ {4d+2k-3 \atop 2d-1} \right\} \\ &\times \left\{ {2d+k-2 \atop d-1} \right\}_2^{-1} \left\{ {2n+2d+2k-3 \atop 2n-1} \right\}^{-1}, \end{aligned}$$

and, for $\lambda = 2$, $r = 0$,

$$l_{n,d} = \left\{ {n-1 \atop d-1} \right\}_2 \left\{ {n \atop d} \right\}_2 \left\{ {n+d+k-1 \atop n-d} \right\}_2^{-1}.$$

**Theorem 2.** *For $1 \le d \le n$ we have*

$$u_{d,n} = \mathbf{i}^{\lambda\frac{k}{2}(1-k)-\lambda k(n+d)+k-kr} q^{\lambda[\frac{k}{2}\left(d+n-\frac{1}{2}+\frac{k}{2}\right)-d+d^2]+\frac{k(r-1)}{2}-r+dr}(1-q)^k \times \frac{(q^\lambda;q^\lambda)_{d+k-2}(q^\lambda;q^\lambda)_{n-1}}{(q^{\lambda(d+k)+r};q^\lambda)_{d-1}(q^{\lambda(n+1)+r};q^\lambda)_{d+k-1}(q^\lambda;q^\lambda)_{n-d}(q^\lambda;q^\lambda)_{k-1}}.$$

Its Fibonacci Corollary:

**Corollary 2.** *For $1 \le d \le n$*

$$u_{d,n} = (-1)^{r(d-1)} \begin{Bmatrix} n+d+k \\ n \end{Bmatrix}_{(r;\lambda)}^{-1} \begin{Bmatrix} d+k-2 \\ d-1 \end{Bmatrix}_{\lambda} \begin{Bmatrix} n-1 \\ d-1 \end{Bmatrix}_{\lambda} \times \left(\prod_{t=1}^{d-1} F_{t\lambda}\right)^2 \left(\prod_{t=0}^{2d+k-2} F_{t\lambda+r}\right)^{-1} F_{\lambda n+r}.$$

From the Corollary above, we give the following examples: for $\lambda = 2$, $r = -1$,

$$u_{d,n} = (-1)^{d-1} \begin{Bmatrix} 2n+2d+2k-3 \\ 2n \end{Bmatrix}^{-1} \begin{Bmatrix} n+d+k-2 \\ n-d \end{Bmatrix}_2 \begin{Bmatrix} 2d+k-2 \\ k-1 \end{Bmatrix}_2 \times \left(\prod_{t=1}^{2d-1} F_{2t}\right)\left(\prod_{t=1}^{2d+k-2} F_{2t-1}\right)^{-1} \frac{1}{F_{2n}},$$

and, for $\lambda = 2$, $r = 0$,

$$u_{d,n} = \begin{Bmatrix} 2d+k-2 \\ d-1 \end{Bmatrix}_2^{-1} \begin{Bmatrix} n-1 \\ d-1 \end{Bmatrix}_2 \begin{Bmatrix} n+d+k-1 \\ n+1 \end{Bmatrix}_2^{-1} \left(\prod_{t=1}^{k-1} F_{2t}\right)^{-1} \frac{1}{F_{2n+2}}.$$

We could also determine the inverses of the matrices $L$ and $U$:

**Theorem 3.** *For $1 \le d \le n$ we have*

$$l_{n,d}^{-1} = \mathbf{i}^{(\lambda k+2)(d-n)} q^{\frac{\lambda}{2}(d-n)(d-k-n+1)} \frac{(q^\lambda;q^\lambda)_{n-1}(q^{\lambda(d+1)+r};q^\lambda)_{n+k-2}}{(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{n-d}(q^{\lambda(n+1)+r};q^\lambda)_{n+k-2}}.$$

Its Fibonacci Corollary:

**Corollary 3.** *For $1 \le d \le n$*

$$l_{n,d}^{-1} = \mathbf{i}^{(d-n)(\lambda+d\lambda-n\lambda+2)} \begin{Bmatrix} n-1 \\ d-1 \end{Bmatrix}_{\lambda} \begin{Bmatrix} n+d+k-1 \\ d+1 \end{Bmatrix}_{(r;\lambda)} \begin{Bmatrix} 2n+k-1 \\ n+1 \end{Bmatrix}_{(r;\lambda)}^{-1}.$$

Thus we have the following examples: for $\lambda = 2$, $r = -1$,

$$l_{n,d}^{-1} = (-1)^{d+n} \begin{Bmatrix} 2n+k-3 \\ n-d \end{Bmatrix}_2 \begin{Bmatrix} 2n-1 \\ 2d-1 \end{Bmatrix} \begin{Bmatrix} 4n+2k-5 \\ 2n-2d \end{Bmatrix}^{-1},$$

and, for $\lambda = 2$, $r = 0$,

$$l_{n,d}^{-1} = (-1)^{d+n} \begin{Bmatrix} n-1 \\ d-1 \end{Bmatrix}_2 \begin{Bmatrix} n+d+k-2 \\ d \end{Bmatrix}_2 \begin{Bmatrix} 2n+k-2 \\ n \end{Bmatrix}_2.$$

**Theorem 4.** *For $1 \le d \le n$ we have*

$$\begin{aligned} u_{d,n}^{-1} &= (-1)^{\frac{\lambda k(d+n)}{2} + \frac{kr}{2} - d + \lambda\frac{k(k-1)}{4} - \frac{k}{2} + n^2} \\ &\quad \times q^{-\lambda\frac{n(n-1)}{2} + r - \lambda\frac{k(d+n)}{2} - \frac{kr}{2} - \lambda nd + \lambda\frac{d(d+1)}{2} - \lambda\frac{k(k-1)}{4} + \frac{k}{2} - rn} \\ &\qquad \times \frac{(q^{\lambda(n+k)+r}; q^\lambda)_n (q^{\lambda(d+1)+r}; q^\lambda)_{n+k-2}}{(q^\lambda; q^\lambda)_{n+k-2} (q^\lambda; q^\lambda)_{d-1} (q^\lambda; q^\lambda)_{n-d}} \frac{(q^\lambda; q^\lambda)_{k-1}}{(1-q)^k}. \end{aligned}$$

And its Fibonacci corollary:

**Corollary 4.** *For $1 \le d \le n$*

$$\begin{aligned} u_{d,n}^{-1} &= (-1)^{n-d+r(1-n)} \mathbf{i}^{n\lambda(1-n) - d\lambda(2n-1-d)} \left( \prod_{t=0}^{2n+k-2} F_{t\lambda+r} \right) \left( \prod_{t=1}^{2n-2} F_{t\lambda} \right) \\ &\quad \times \begin{Bmatrix} 2n+k \\ n \end{Bmatrix}_{(r,\lambda)} \begin{Bmatrix} n+d+k-1 \\ d+1 \end{Bmatrix}_{(r,\lambda)} \begin{Bmatrix} 2n+k-2 \\ n \end{Bmatrix}_{(r,\lambda)}^{-1} \\ &\qquad \times \begin{Bmatrix} n+k-2 \\ k-1 \end{Bmatrix}_{\lambda}^{-1} \begin{Bmatrix} n-1 \\ d-1 \end{Bmatrix}_{\lambda} \begin{Bmatrix} 2n-2 \\ n-1 \end{Bmatrix}_{\lambda}. \end{aligned}$$

Especially for $\lambda = 2$, $r = -1$,

$$\begin{aligned} u_{d,n}^{-1} &= (-1)^{d+1} \begin{Bmatrix} 2n+2d+2k-5 \\ 2d-2 \end{Bmatrix} \begin{Bmatrix} 2n+k-3 \\ n-d \end{Bmatrix}_2 \begin{Bmatrix} 2n+k-3 \\ k-1 \end{Bmatrix}_2^{-1} \\ &\quad \times \left( \prod_{t=1}^{2n+k-1} F_{2t-1} \right) \left( \prod_{t=1}^{2n-2} F_{2t} \right)^{-1} \frac{1}{F_{2d-1}}, \end{aligned}$$

and, for $\lambda = 2$, $r = 0$,

$$u_{d,n}^{-1} = (-1)^{d+n} \begin{Bmatrix} n+d+k-2 \\ d \end{Bmatrix}_2 \begin{Bmatrix} 2n+k-1 \\ n \end{Bmatrix}_2 \begin{Bmatrix} n \\ d-1 \end{Bmatrix}_2 \left( \prod_{t=1}^{k-1} F_{2t} \right) F_{2d}.$$

As a consequence, we can compute the determinant of $\mathcal{Q}_n$, since it is simply evaluated as $u_{1,1} \cdots u_{n,n}$ (we only state the Fibonacci versions):

**Theorem 5.**

$$\det \mathcal{G}_n = (-1)^{\frac{r}{2}n(n-1)} \prod_{d=1}^{n} \begin{Bmatrix} 2d+k \\ d \end{Bmatrix}_{(r,\lambda)}^{-1} \begin{Bmatrix} d+k-2 \\ d-1 \end{Bmatrix}_{\lambda} \times \Big(\prod_{t=1}^{d-1} F_{t\lambda}\Big)^2 \Big(\prod_{t=0}^{2d+k-2} F_{t\lambda+r}\Big)^{-1} F_{\lambda d+r}.$$

As examples, we have that for $\lambda = 2$ and $r = -1$,

$$\det \mathcal{G}_n = (-1)^{\frac{1}{2}n(n+3)} \prod_{d=1}^{n} \begin{Bmatrix} 4d+2k-3 \\ 2d \end{Bmatrix}^{-1} \begin{Bmatrix} 2d+k-2 \\ k-1 \end{Bmatrix}_2 \times \Big(\prod_{t=1}^{2d-1} F_{2t}\Big)\Big(\prod_{t=1}^{2d+k-2} F_{2t-1}\Big)^{-1} \frac{1}{F_{2d}},$$

and, for $\lambda = 2$, $r = -1$

$$\det \mathcal{G}_n = \Big(\prod_{v=1}^{k-1} F_{2v}\Big)^{-1} \prod_{d=1}^{n} \begin{Bmatrix} 2d+k-2 \\ d-1 \end{Bmatrix}_2^{-1} \begin{Bmatrix} 2d+k-1 \\ d+1 \end{Bmatrix}_2^{-1} \frac{1}{F_{2d+2}}.$$

Now we compute the inverse of the matrix $\mathcal{G}$. This time it depends on the dimension, so we compute $(\mathcal{G}_n)^{-1}$.

**Theorem 6.** *For $1 \le i, j \le n$:*

$$\begin{aligned} &\big((\mathcal{G}_n)^{-1}\big)_{i,k} \\ &= (-1)^{(j-i)-\frac{k}{2}(1-r)-\left(\frac{1-k}{2}-i-j\right)\frac{k\lambda}{2}} q^{r-\left(1-i-j-j^2\right)\frac{\lambda}{2}+\left(\frac{1-k}{2}-i-j\right)\frac{k\lambda}{2}+\frac{k}{2}(1-r)} \\ &\quad\times \frac{\left(q^\lambda;q^\lambda\right)_{k-1}}{(1-q)^k\,(q^\lambda;q^\lambda)_{j-1}(q^\lambda;q^\lambda)_{i-1}\,(q^r;q^\lambda)_{i+1}\,(q^r;q^\lambda)_{j+1}} \\ &\quad\times \sum_{\max\{i,j\}\le h\le n} \frac{\left(q^r;q^\lambda\right)_{h+k+i-1}(q^r;q^\lambda)_{h+1}(q^r;q^\lambda)_{h+k+j-1}(q^\lambda;q^\lambda)_{h-1}}{\left(q^r;q^\lambda\right)_{h+k}(q^\lambda;q^\lambda)_{h+k-2}(q^\lambda;q^\lambda)_{h-i}(q^\lambda;q^\lambda)_{h-j}} \\ &\quad\times \left(1-q^{\lambda(2h+k-1)+r}\right) q^{-hj\lambda-hr-ih\lambda}. \end{aligned}$$

Finally, we provide the Cholesky decomposition.

**Theorem 7.** *For $i,j\geq 1$:*

$$\mathcal{C}_{i,j}=\frac{(q^{\lambda};q^{\lambda})_{i-1}(1-q)^{\frac{k}{2}}}{(q^{\lambda(i+1)+r};q^{\lambda})_{j+k-1}(q^{\lambda};q^{\lambda})_{i-j}}$$
$$\times\,\mathbf{i}^{-\lambda\frac{k^2}{4}+\lambda\frac{k}{4}+\frac{k}{2}+\frac{3rk}{2}-\lambda ik}q^{\lambda\frac{j(j-1)}{2}+\lambda\frac{ki}{2}+\lambda\frac{k^2}{8}-\lambda\frac{k}{8}-\frac{k}{4}+\frac{rj}{2}+\frac{kr}{4}-\frac{r}{2}}$$
$$\times\sqrt{\frac{(1-q^{\lambda(2j+k-1)+r})(q^{\lambda};q^{\lambda})_{j+k-2}(q^{\lambda(j+1)+r};q^{\lambda})_{k-1}}{(q^{\lambda};q^{\lambda})_{k-1}(q^{\lambda};q^{\lambda})_{j-1}}}.$$

Its Fibonacci Corollary:

**Corollary 5.** *For $i,j\geq 1$:*

$$\mathcal{C}_{i,j}=\mathbf{i}^{(j\lambda+r)(j-1)}\,(-1)^{kr}\begin{Bmatrix}i+j+k\\ i+1\end{Bmatrix}_{(r,\lambda)}^{-1}\begin{Bmatrix}i-1\\ j-1\end{Bmatrix}_{\lambda}\left(\prod_{t=0}^{j+k-2}F_{\lambda t+r}\right)^{-1}$$
$$\times\left(\prod_{t=1}^{j-1}F_{\lambda t}\right)\sqrt{\begin{Bmatrix}j+k-2\\ k-1\end{Bmatrix}_{\lambda}\begin{Bmatrix}j+k\\ j+1\end{Bmatrix}_{(r,\lambda)}\left(\prod_{t=0}^{k-2}F_{\lambda t+r}\right)F_{\lambda(2j+k-1)+r}}.$$

From the Corollary above, we give the following examples: for $\lambda=2$, $r=-1$,

$$\mathcal{C}_{i,j}=\mathbf{i}^{1-j}\,(-1)^{k}\begin{Bmatrix}i+j+k-1\\ i\end{Bmatrix}_{(1,2)}^{-1}\begin{Bmatrix}i-1\\ j-1\end{Bmatrix}_{2}\left(\prod_{t=1}^{j-1}F_{2t}\right)$$
$$\times\sqrt{\begin{Bmatrix}j+k-2\\ k-1\end{Bmatrix}_{2}\begin{Bmatrix}2j+k-1\\ j\end{Bmatrix}_{(1,2)}\left(\prod_{t=1}^{2j+k-1}F_{2t-1}\right)^{-1}}$$

and, for $\lambda=2$, $r=0$,

$$\mathcal{C}_{i,j}=(-1)^{j(j-1)}\begin{Bmatrix}i+j+k-1\\ i\end{Bmatrix}_{2}^{-1}\begin{Bmatrix}i-1\\ j-1\end{Bmatrix}_{2}\sqrt{\frac{F_{2(2j+k-1)}}{F_{2j}F_{2(j+k-1)}}\left(\prod_{t=1}^{k-1}F_{2t}\right)^{-1}}.$$

## 2. Proofs

We compute

$$\sum_d l_{md}u_{dn}$$
$$=\sum_d\mathbf{i}^{\lambda k(d-m)}q^{\lambda\frac{k(m-d)}{2}}\frac{(q^{\lambda};q^{\lambda})_{m-1}(q^{\lambda(d+1)+r};q^{\lambda})_{d+k-1}}{(q^{\lambda};q^{\lambda})_{d-1}(q^{\lambda};q^{\lambda})_{m-d}(q^{\lambda(m+1)+r};q^{\lambda})_{d+k-1}}$$
$$\times\,\mathbf{i}^{\lambda\frac{k}{2}(1-k)-\lambda k(n+d)+k-kr}q^{\lambda[\frac{k}{2}\left(d+n-\frac{1}{2}+\frac{k}{2}\right)-d+d^2]+\frac{k(r-1)}{2}-r+dr}(1-q)^k$$
$$\times\frac{(q^{\lambda};q^{\lambda})_{d+k-2}(q^{\lambda};q^{\lambda})_{n-1}}{(q^{\lambda(d+k)+r};q^{\lambda})_{d-1}(q^{\lambda(n+1)+r};q^{\lambda})_{d+k-1}(q^{\lambda};q^{\lambda})_{n-d}(q^{\lambda};q^{\lambda})_{k-1}}.$$

From this, we only continue with terms that depend on the summation index $d$:

$$\sum_d q^{\lambda(-d+d^2)+dr}\frac{(q^r;q^\lambda)_{2d+k}}{(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{m-d}(q^r;q^\lambda)_{m+d+k}}\\ \times\frac{(q^\lambda;q^\lambda)_{d+k-2}}{(q^r;q^\lambda)_{2d+k-1}(q^r;q^\lambda)_{n+d+k}(q^\lambda;q^\lambda)_{n-d}}.$$

We set $Q:=q^\lambda$ and $s=r/\lambda$:

$$\sum_d Q^{-d+d^2+ds}\frac{(Q^s;Q)_{2d+k}}{(Q;Q)_{d-1}(Q;Q)_{m-d}(Q^s;Q)_{m+d+k}}\\ \times\frac{(Q;Q)_{d+k-2}}{(Q^s;Q)_{2d+k-1}(Q^s;Q)_{n+d+k}(Q;Q)_{n-d}}.$$

Apart from a constant factor, this is the sum that has been evaluated already in [2], when $(q,r)$ from [2] is replaced by $(Q,s)$.

Now we look at the inverse matrices:

$$\begin{aligned}
&\sum_{n\le d\le m} l_{m,d}l^{-1}_{d,n}\\
&=\sum_{n\le d\le m}\mathbf{i}^{\lambda k(d-m)}q^{\lambda\frac{k(m-d)}{2}}\frac{(q^\lambda;q^\lambda)_{m-1}(q^{\lambda(d+1)+r};q^\lambda)_{d+k-1}}{(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{m-d}(q^{\lambda(m+1)+r};q^\lambda)_{d+k-1}}\\
&\times\mathbf{i}^{(\lambda k+2)(n-d)}q^{\frac{\lambda}{2}(n-d)(n-k-d+1)}\frac{(q^\lambda;q^\lambda)_{d-1}(q^{\lambda(n+1)+r};q^\lambda)_{d+k-2}}{(q^\lambda;q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{d-n}(q^{\lambda(d+1)+r};q^\lambda)_{d+k-2}}\\
&=\mathbf{i}^{\lambda k(n-m)}\sum_{n\le d\le m}q^{\lambda\frac{k(m-d)}{2}}\frac{(q^\lambda;q^\lambda)_{m-1}(q^{\lambda(d+1)+r};q^\lambda)_{d+k-1}}{(q^\lambda;q^\lambda)_{m-d}(q^{\lambda(m+1)+r};q^\lambda)_{d+k-1}}\\
&\times(-1)^{n-d}q^{\frac{\lambda}{2}(n-d)(n-k-d+1)}\frac{(q^{\lambda(n+1)+r};q^\lambda)_{d+k-2}}{(q^\lambda;q^\lambda)_{n-1}(q^\lambda;q^\lambda)_{d-n}(q^{\lambda(d+1)+r};q^\lambda)_{d+k-2}}.
\end{aligned}$$

We only continue with terms that depend on the summation index $d$:

$$\sum_{n\le d\le m}\frac{(-1)^d q^{-\lambda nd+\frac{\lambda}{2}d(d-1)}(q^{\lambda(d+1)+r};q^\lambda)_{d+k-1}(q^{\lambda(n+1)+r};q^\lambda)_{d+k-2}}{(q^\lambda;q^\lambda)_{m-d}(q^{\lambda(m+1)+r};q^\lambda)_{d+k-1}(q^\lambda;q^\lambda)_{d-n}(q^{\lambda(d+1)+r};q^\lambda)_{d+k-2}}.$$

We replace $Q:=q^\lambda$, $s:=r/\lambda$ and leave out irrelevant factors:

$$\sum_{n\le d\le m}\frac{(-1)^d Q^{-nd+\binom{d}{2}}(1-Q^{s+2d+k-1})(Q^s;Q)_{n+d+k-1}}{(Q;Q)_{m-d}(Q;Q)_{d-n}(Q^s;Q)_{m+d+k}}.$$

Apart from a constant factor, this is the sum that has been evaluated already in [2], when $(q,r)$ from [2] is replaced by $(Q,s)$.

$$
\begin{aligned}
&\sum_{m\le d\le n} u_{m,d}u^{-1}_{d,n}\\
&=\sum_{m\le d\le n} \mathbf{i}^{\lambda\frac{k}{2}(1-k)-\lambda k(d+m)+k-kr} q^{\lambda[\frac{k}{2}\left(m+d-\frac12+\frac{k}{2}\right)-m+m^2]+\frac{k(r-1)}{2}-r+mr}(1-q)^k\\
&\times \frac{(q^\lambda;q^\lambda)_{m+k-2}(q^\lambda;q^\lambda)_{d-1}}{(q^{\lambda(m+k)+r};q^\lambda)_{m-1}(q^{\lambda(d+1)+r};q^\lambda)_{m+k-1}(q^\lambda;q^\lambda)_{d-m}(q^\lambda;q^\lambda)_{k-1}}\\
&\times (-1)^{\frac{\lambda k(d+n)}{2}+\frac{kr}{2}-d+\lambda\frac{k(k-1)}{4}-\frac{k}{2}+n^2}\\
&\times q^{-\lambda\frac{n(n-1)}{2}+r-\lambda\frac{k(d+n)}{2}-\frac{kr}{2}-\lambda nd+\lambda\frac{d(d+1)}{2}-\lambda\frac{k(k-1)}{4}+\frac{k}{2}-rn}\\
&\times \frac{(q^{\lambda(n+k)+r};q^\lambda)_n(q^{\lambda(d+1)+r};q^\lambda)_{n+k-2}}{(q^\lambda;q^\lambda)_{n+k-2}(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{n-d}}\frac{(q^\lambda;q^\lambda)_{k-1}}{(1-q)^k}.
\end{aligned}
$$

Once again, we only write the terms that do depend on $d$:

$$
\sum_{m\le d\le n}\frac{(-1)^d\,q^{-\lambda nd+\lambda\frac{d(d+1)}{2}}(q^\lambda;q^\lambda)_{d-1}}{(q^{\lambda(d+1)+r};q^\lambda)_{m+k-1}(q^\lambda;q^\lambda)_{d-m}}\frac{(q^{\lambda(d+1)+r};q^\lambda)_{n+k-2}}{(q^\lambda;q^\lambda)_{n+k-2}(q^\lambda;q^\lambda)_{d-1}(q^\lambda;q^\lambda)_{n-d}}.
$$

And again we do the usual replacement and ignore irrelevant factors:

$$
\sum_{m\le d\le n}\frac{(-1)^d\,Q^{-nd+\frac{d(d+1)}{2}}(Q^s;Q)_{d+n+k-1}}{(Q^s;Q)_{d+m+k}(Q;Q)_{d-m}(Q;Q)_{n+k-2}(Q;Q)_{n-d}}.
$$

And once again, this has been evaluated already in our previous paper.

Finally, for the Cholesky decomposition, we need to consider

$$
\sum_{1\le j\le\min\{i,l\}}\mathcal{C}_{i,j}\mathcal{C}_{l,j},
$$

or

$$
\begin{aligned}
&\sum_{1\le j\le\min\{i,l\}}\frac{(q^\lambda;q^\lambda)_{i-1}(1-q)^{\frac{k}{2}}}{(q^{\lambda(i+1)+r};q^\lambda)_{j+k-1}(q^\lambda;q^\lambda)_{i-j}}q^{\lambda\frac{j(j-1)}{2}+\lambda\frac{ki}{2}+\lambda\frac{k^2}{8}-\lambda\frac{k}{8}-\frac{k}{4}+\frac{rj}{2}+\frac{kr}{4}-\frac{r}{2}}\\
&\times \mathbf{i}^{-\lambda\frac{k^2}{4}+\lambda\frac{k}{4}+\frac{k}{2}+\frac{3rk}{2}-\lambda ik}\sqrt{\frac{(1-q^{\lambda(2j+k-1)+r})(q^\lambda;q^\lambda)_{j+k-2}(q^{\lambda(j+1)+r};q^\lambda)_{k-1}}{(q^\lambda;q^\lambda)_{k-1}(q^\lambda;q^\lambda)_{j-1}}}\\
&\times \frac{(q^\lambda;q^\lambda)_{l-1}}{(q^{\lambda(l+1)+r};q^\lambda)_{j+k-1}(q^\lambda;q^\lambda)_{l-j}}(1-q)^{\frac{k}{2}}q^{\lambda\frac{j(j-1)}{2}+\lambda\frac{kl}{2}+\lambda\frac{k^2}{8}-\lambda\frac{k}{8}-\frac{k}{4}+\frac{rj}{2}+\frac{kr}{4}-\frac{r}{2}}\\
&\times \mathbf{i}^{-\lambda\frac{k^2}{4}+\lambda\frac{k}{4}+\frac{k}{2}+\frac{3rk}{2}-\lambda lk}\sqrt{\frac{(1-q^{\lambda(2j+k-1)+r})(q^\lambda;q^\lambda)_{j+k-2}(q^{\lambda(j+1)+r};q^\lambda)_{k-1}}{(q^\lambda;q^\lambda)_{k-1}(q^\lambda;q^\lambda)_{j-1}}}
\end{aligned}
$$

We only let the terms survive that do depend on the summation index $j$:

$$\sum_{1\le j\le \min\{i,l\}} \frac{q^{\lambda j(j-1)+rj}}{(q^{\lambda(i+1)+r};q^{\lambda})_{j+k-1}(q^{\lambda};q^{\lambda})_{i-j}} \times \frac{(1-q^{\lambda(2j+k-1)+r})(q^{\lambda};q^{\lambda})_{j+k-2}(q^{\lambda(j+1)+r};q^{\lambda})_{k-1}}{(q^{\lambda};q^{\lambda})_{j-1}(q^{\lambda(l+1)+r};q^{\lambda})_{j+k-1}(q^{\lambda};q^{\lambda})_{l-j}}.$$

Rewriting it:

$$\sum_{1\le j\le \min\{i,l\}} \frac{Q^{j(j-1)+sj}(1-Q^{2j+k+s-1})(Q;Q)_{j+k-2}(Q^s;Q)_{j+k}}{(Q^s;Q)_{i+j+k}(Q;Q)_{i-j}(Q^s;Q)_{j+1}(Q;Q)_{j-1}(Q^s;Q)_{l+j+k}(Q;Q)_{l-j}}.$$

And this is again the sum already studied in our previous paper.

## References


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TOBB University of Economics and Technology Mathematics Department 06560 Ankara Turkey

*E-mail address*: ekilic@etu.edu.tr

Department of Mathematics, University of Stellenbosch 7602 Stellenbosch South Africa

*E-mail address*: hproding@sun.ac.za