Generalized Binary Recurrent Quasi-Cyclic Matrices

E. Kılıç^a, Y.T. Ulutaş^b, I. Akkus^{c*}, N. Ömür^b

^{a}TOBB University of Economics and Technology Mathematics Department 06560
Ankara Turkey
`ekilic@etu.edu.tr`

^{b}Kocaeli University Mathematics Department 41380 Izmit Kocaeli Turkey
`turkery@kocaeli.edu.tr`,`neseomur@kocaeli.edu.tr`

^{c}Kırıkkale University, Faculty of Arts and Sciences, Department of Mathematics, 71450
Yahsihan, Kırıkkale, Turkey
`iakkus.tr@gmail.com`

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Abstract

In this paper, we obtain solutions to infinite family of Pell equations of higher degree based on the more generalized Fibonacci and Lucas sequences as well as their all subsequences of the form $\{u_{kn}\}$ and $\{v_{kn}\}$ for odd $k > 0$.

Keywords: Quasi-cyclic matrices, binary linear recurrences, Pell equation.

MSC: 11B37, 15A15.

1. Introduction

The generalized Fibonacci and Lucas sequences are defined by

$$u_{n+1} = Au_n + Bu_{n-1} \tag{1.1}$$

and

$$v_{n+1} = Av_n + Bv_{n-1}, \tag{1.2}$$

**Corresponding author*

where $u_0 = 0$, $u_1 = 1$ and $v_0 = 2$, $v_1 = A$, respectively.

For $k \geq 0$ and $n > 1$, the sequences $\{u_{kn}\}$ and $\{v_{kn}\}$ satisfy the recursions (see [1]):

$$u_{kn} = v_k u_{k(n-1)} - (-B)^k u_{k(n-2)} \text{ and } v_{kn} = v_k v_{k(n-1)} - (-B)^k v_{k(n-2)}. \quad (1.3)$$

The Binet formulae are

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } v_n = \alpha^n + \beta^n,$$

where $\alpha, \beta = A \pm \sqrt{A^2 + 4B}$.

By the Binet formulae note that for a fixed $k > 0$,

$$u_{-kn} = (-1)^{kn+1} u_{kn} \text{ and } u_{2kn} = v_{kn} u_{kn}. \quad (1.4)$$

A $n \times n$ quasi-cyclic matrix $R(D; x_1, x_2, ...x_n)$ (or shortly R) has the form (see [2, 4, 5]):

$$R = \begin{pmatrix} x_1 & Dx_n & Dx_{n-1} & ... & Dx_3 & Dx_2 \\ x_2 & x_1 & Dx_n & ... & Dx_4 & Dx_3 \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ x_{n-1} & x_{n-2} & x_{n-3} & ... & x_1 & Dx_n \\ x_n & x_{n-1} & x_{n-2} & ... & x_2 & x_1 \end{pmatrix}.$$

The classical Pell equation $x^2 - dy^2 = \pm 1$ ($d \in Z$) can be rewritten as

$$\det \begin{pmatrix} x & dy \\ y & x \end{pmatrix} = \pm 1.$$

By means of quasi-cyclic determinants, the equation

$$\det \begin{pmatrix} x_1 & Dx_n & Dx_{n-1} & ... & Dx_3 & Dx_2 \\ x_2 & x_1 & Dx_n & ... & Dx_4 & Dx_3 \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ x_{n-1} & x_{n-2} & x_{n-3} & ... & x_1 & Dx_n \\ x_n & x_{n-1} & x_{n-2} & ... & x_2 & x_1 \end{pmatrix} = \pm 1$$

is called Pell's equation of degree n.

In [2], the author gave a method to generalize the classical Pell equation whose degree is $n = 2$ to a Pell equation of degree $n \geq 2$ by some $n \times n$ quasi-cyclic determinants. In particular, the author proved that for $n \geq 2$,

$$\det(R(L_n; F_{2n-1}, F_{2n-2}, ..., F_n)) = 1, \quad (1.5)$$

where L_n and F_n denote the nth Lucas and Fibonacci number, respectively. Further it was showed that

$$\det(R(L_n; F_{2n-1+k}, F_{2n-2+k}, ..., F_{n+k})) = (-1)^{n-1} L_n F_k^n + F_{k-1}^n,$$

where k is an integer.

In [3], the author generalized the results given in [2] by giving a relationship between certain Pell equations of degree n and general Fibonacci and Lucas sequences. For example, for $k = 1$ in (1.3) and (1.4) and $n > 1$, we have

$$\det\left(R\left(v_n; u_{2n-1}, u_{2n-2}, ..., u_n\right)\right) = B^{n(n-1)}, \tag{1.6}$$

where B is defined as before.

From [4, 5], the following two propositions are known:

Proposition 1. *For $n > 0$,*

$$\det\left(R\right) = \prod_{k=0}^{n-1}\left(\sum_{i=1}^{n} x_i d^{i-1}\varepsilon^{k(i-1)}\right), \tag{1.7}$$

where $d = \sqrt[n]{D}$, $\varepsilon = e^{2\pi i/n}$ and each factor $\sum_{i=1}^{n} x_i d^{i-1}\varepsilon^{k(i-1)}$ of the RHS of (1.7) is an eigenvalue of the matrix R.

Proposition 2. *Let n and D be fixed. Then the sum, differences, and product of two quasi-cyclic matrices is also quasi-cyclic. The inverse of a quasi-cyclic matrix is quasi-cyclic.*

In this paper, we generalize the results of [2, 3] and so obtain solutions to infinite family of Pell equations of higher degree based on more generalized Fibonacci and Lucas sequences as well as their all subsequences of the form $\{u_{kn}\}$ and $\{v_{kn}\}$, for odd $k > 0$.

2. Quasi-cyclic matrices via the generalized Fibonacci and Lucas numbers

We obtain some results about infinite family of Pell equations of higher degree by using certain quasi-cyclic determinants with the generalized Fibonacci and Lucas numbers. We give some auxiliary results for further use and denote $(-B)^k$ by b for easy writing.

Lemma 2.1. *For positive integers k and n,*

$$v_k u_{k(2n-1)} - v_{kn}u_{kn} = bu_{k(2n-2)},$$
$$b\left(u_{k(2n-1)} - v_{kn}u_{k(n-1)}\right) = b^n u_k,$$
$$u_{kn}^2 - u_{k(n+1)}u_{k(n-1)} = b^{(n-1)}u_k^2.$$

Proof. The claimed identities follows from the Binet formulae. $\qquad\square$

Theorem 2.2. *For $n \geq 2$,*

$$\det\left(R\left(v_{kn}; u_{k(2n-1)}, u_{k(2n-2)}, ..., u_{kn}\right)\right) = b^{n(n-1)}u_k^n. \tag{2.1}$$

Proof. For $n = 2$,

$$\det \left(R \left(v_{2k}; u_{3k}, u_{2k} \right) \right) = \begin{vmatrix} u_{3k} & v_{2k} u_{2k} \\ u_{2k} & u_{3k} \end{vmatrix} = u_{3k}^2 - v_{2k} u_{2k}^2 = b^2 u_k^2.$$

For $n > 2$, consider the upper triangular matrix

$$T = \begin{pmatrix} 1 & -v_k & b & & 0 \\ & 1 & -v_k & \ddots & \\ & & \ddots & \ddots & b \\ & & & 1 & -v_k \\ & & & & 1 \end{pmatrix}. \tag{2.2}$$

From a matrix multiplication and by Lemma 2.1, we get

$$RT = \begin{pmatrix} u_{k(2n-1)} & -bu_{k(2n-2)} & b^n u_k & 0 & \dots & 0 \\ u_{k(2n-2)} & -bu_{k(2n-3)} & 0 & b^n u_k & \ddots & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & b^n u_k \\ u_{k(n+1)} & -bu_{kn} & 0 & 0 & \dots & 0 \\ u_{kn} & -bu_{k(n-1)} & 0 & 0 & \dots & 0 \end{pmatrix} \tag{2.3}$$

Then we write

$$
\begin{aligned}
\det R \;&=\; (\det R)(\det T) = \det (RT) \\
&=\; \left(bu_{kn}^2 - bu_{k(n+1)}u_{k(n-1)} \right) \det \begin{pmatrix} b^n u_k & 0 & \cdots & 0 \\ 0 & b^n u_k & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & b^n u_k \end{pmatrix} \\
&=\; \left(bu_{kn}^2 - bu_{k(n+1)}u_{k(n-1)} \right) \left(b^n u_k \right)^{n-2} \\
&=\; b^{n(n-1)} u_k^n,
\end{aligned}
$$

as claimed. $\qquad\square$

Corollary 2.3. *For $n \geq 2$,*

$$\prod_{k=0}^{n-1} \left(\sum_{j=1}^{n} u_{k(2n-j)} \left(\sqrt[n]{v_{kn}} \right)^{j-1} \varepsilon^{k(j-1)} \right) = b^{n(n-1)} u_k^n,$$

where $\sqrt[n]{v_{kn}}$ is the nth complex root of v_{kn} and $\varepsilon = e^{2\pi i/n}$.

We shall need the following identities:

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1. $-bu_{k(2n-3)} + v_k u_{k(2n-2)} - u_{k(2n-1)} = 0, ..., -bu_{kn} + v_k u_{k(n+1)} - u_{k(n+2)} = 0$,

2. $u_{k(2n-1)} - v_{kn} u_{k(n-1)} = b^{n-1} u_k$,

3. $E_n^{n+1} = v_{kn} E_n$ and $E_n^n = v_{kn} I_n$, where

$$
E_n = \begin{pmatrix}
0 & 0 & \cdots & 0 & v_{kn} \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}.
$$

Theorem 2.4. *For $n \geq 3$, the matrix $R\left(v_{kn}; u_{k(2n-1)}, u_{k(2n-2)}, ..., u_{kn}\right)$ is invertible and its inverse matrix R^{-1} is given by*

$$
R^{-1}\left(v_{kn}; u_{k(2n-1)}, u_{k(2n-2)}, ..., u_{kn}\right) = -\frac{1}{u_k b^n}\left(-bI_n + v_k E_n - E_n^2\right), \qquad (2.4)
$$

where I_n is the $n \times n$ identity matrix and the matrix E_n is defined as before.

Proof. Since $\det\left(R\left(v_{kn}; u_{k(2n-1)}, u_{k(2n-2)}, ..., u_{kn}\right)\right) \neq 0$ by Theorem 2.2, its inverse exists. It is easy to see that

$$
R\left(v_{kn}; u_{k(2n-1)}, u_{k(2n-2)}, ..., u_{kn}\right) = \left(u_{k(2n-1)} I_n + u_{k(2n-2)} E_n + ... + u_{kn} E_n^{n-1}\right).
$$

Hence,

$$
\begin{aligned}
& R\left(v_{kn}; u_{k(2n-1)}, u_{k(2n-2)}, ..., u_{kn}\right) R^{-1}\left(v_{kn}; u_{k(2n-1)}, u_{k(2n-2)}, ..., u_{kn}\right) \\
=\ & \left(u_{k(2n-1)} I_n + u_{k(2n-2)} E_n + ... + u_{kn} E_n^{n-1}\right)\left(\frac{-1}{u_k b^n}\right)\left(-(-B)^k I_n + v_k E_n - E_n^2\right) \\
=\ & \left(-bu_{k(2n-1)} I_n + \left(u_{2kn} - u_{kn} v_{kn}\right) E_n + \left(v_k u_{kn} - u_{k(n+1)}\right) v_{kn} I_n\right)\left(\frac{-1}{u_k b^n}\right) \\
=\ & -b\left(u_{k(2n-1)} - v_{kn} u_{k(n-1)}\right) I_n\left(\frac{-1}{u_k b^n}\right) \\
=\ & -b\left(b^{(n-1)} u_k\right) I_n\left(\frac{-1}{u_k b^n}\right) \\
=\ & I_n,
\end{aligned}
$$

as claimed. $\qquad\square$

3. The Determinants of Quasi-Cyclic Matrices

For all integer t, define the $n \times n$ quasi-cyclic matrix $R_{k,n,t}$ as

$$
R_{k,n,t} = R\left(v_{kn}; u_{k(2n-1+t)}, u_{k(2n-2+t)}, ..., u_{k(n+t)}\right).
$$

By Theorem 2.2, we have

$$\det\left(R_{k,n,0}\right) = b^{n(n-1)}u_k^n.$$

For $\det R_{k,n,1}$, $\det R_{k,n,2}$,..., $\det R_{k,n,-1}$, $\det R_{k,n,-2}$,..., we can obtain corresponding results.

Define the $n \times n$ matrices $g_{k,n,t}$ and $h_{k,n,t}$ as shown:

$$g_{k,n,t} = \begin{pmatrix} u_{k(2n+t-1)} & -bu_{k(2n+t-2)} & -b^{n+1}u_{k(t-1)} & & 0 \\ u_{k(2n+t-2)} & -bu_{k(2n+t-3)} & b^n u_{kt} & \ddots & \\ \vdots & \vdots & 0 & \ddots & -b^{n+1}u_{k(t-1)} \\ u_{k(n+t+1)} & -bu_{k(n+t)} & \vdots & \ddots & b^n u_{kt} \\ u_{k(n+t)} & -bu_{k(n+t-1)} & 0 & \dots & 0 \end{pmatrix}$$

and

$$h_{k,n,t} = \begin{pmatrix} u_{k(2n+t-1)} & b^n u_{kt} & -b^{n+1}u_{k(t-1)} & & & 0 \\ u_{k(2n+t-2)} & 0 & b^n u_{kt} & -b^{n+1}u_{k(t-1)} & & \\ \vdots & \vdots & 0 & b^n u_{kt} & \ddots & \\ \vdots & \vdots & \vdots & 0 & \ddots & -b^{n+1}u_{k(t-1)} \\ u_{k(n+t+1)} & 0 & 0 & \dots & \ddots & b^n u_{kt} \\ u_{k(n+t)} & 0 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

We give some auxiliary Lemmas before the proof of main Theorem.

Lemma 3.1. *(The recurrence of $\det g_{k,n,t}$)*

$$\det g_{k,n,t} = (-1)^n b^{\left(n^2-n+t\right)}u_k u_{k(n-1)}u_{kt}^{n-2} - b^{(2n-1)}u_{k(t-1)}\det g_{k,n-1,t}. \quad (3.1)$$

Proof. Clearly

$\det g_{k,n,t}$

$$= -b^{n(n-2)+1}\begin{vmatrix} u_{k(2n+t-1)} & u_{k(2n+t-2)} & -bu_{k(t-1)} & 0 & \dots & 0 \\ u_{k(2n+t-2)} & u_{k(2n+t-3)} & u_{kt} & -bu_{k(t-1)} & \ddots & \vdots \\ \vdots & \vdots & 0 & u_{kt} & \ddots & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & -bu_{k(t-1)} \\ u_{k(n+t+1)} & u_{k(n+t)} & \vdots & \vdots & \ddots & u_{kt} \\ u_{k(n+t)} & u_{k(n+t-1)} & 0 & \dots & \dots & 0 \end{vmatrix}.$$

By subtracting the second column of $g_{k,n,t}$ from the first column by multiplying v_k gives us

$\det g_{k,n,t}$

$$= -b^{n(n-2)+1} \begin{vmatrix} bu_{k(2n+t-3)} & u_{k(2n+t-2)} & -bu_{k(t-1)} & 0 & \cdots & 0 \\ bu_{k(2n+t-4)} & u_{k(2n+t-3)} & u_{kt} & -bu_{k(t-1)} & \ddots & \vdots \\ \vdots & \vdots & 0 & u_{kt} & \ddots & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & -bu_{k(t-1)} \\ u_{k(n+t-1)} & u_{k(n+t)} & \vdots & \vdots & \ddots & u_{kt} \\ bu_{k(n+t-2)} & u_{k(n+t-1)} & 0 & \cdots & \cdots & 0 \end{vmatrix} . \quad \blacksquare$$

So on after $n+t-1$ subtractions between the two columns, we get finally

$\det g_{k,n,t}$

$$= -b^{n(n-2)+n+t} \begin{vmatrix} u_{kn} & u_{k(n-1)} & -bu_{k(t-1)} & 0 & \cdots & 0 \\ u_{k(n-1)} & u_{k(n-2)} & u_{kt} & -bu_{k(t-1)} & \ddots & \vdots \\ \vdots & \vdots & 0 & u_{kt} & \ddots & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & -bu_{k(t-1)} \\ u_{2k} & u_1 & \vdots & \vdots & \ddots & u_{kt} \\ u_k & u_0 & 0 & \cdots & \cdots & 0 \end{vmatrix} .$$

Expanding the determinant above with respect to the first row and by $u_0 = 0$, we get

$$\det g_{k,n,t} = b^{\left(n^2-n+t\right)} u_{k(n-1)} \begin{vmatrix} u_{k(n-1)} & u_{kt} & \cdots & \vdots \\ \vdots & 0 & \cdots & -bu_{k(t-1)} \\ \vdots & \vdots & \cdots & u_{kt} \\ u_k & 0 & \cdots & 0 \end{vmatrix}$$

$$+ b^{n^2-n+t+1} u_{k(t-1)} \begin{vmatrix} u_{k(n-1)} & u_{k(n-2)} & -bu_{k(t-1)} & 0 & 0 \\ u_{k(n-2)} & u_{k(n-3)} & u_{kt} & \cdots & \vdots \\ \vdots & \vdots & 0 & \cdots & -bu_{k(t-1)} \\ \vdots & \vdots & \vdots & \cdots & u_{kt} \\ u_k & u_0 & 0 & \cdots & 0 \end{vmatrix}$$

$$= (-1)^n b^{n^2-n+t} u_{k(n-1)} u_k u_{kt}^{n-2} + b^{n^2-n+t+1} u_{k(t-1)} \left(\frac{-1}{b^{n^2-3n+t+2}} \right) \det g_{k,n-1,t}$$

$$= (-1)^n b^{n^2-n+t} u_{k(n-1)} u_k u_{kt}^{n-2} - b^{2n-1} u_{k(t-1)} \det g_{k,n-1,t}. \quad \blacksquare$$

Thus we have the conclusion. $\qquad\square$

Lemma 3.2. *For odd $k > 0$,*

$$\det g_{k,n,t} = \frac{(-1)^{kn}}{u_k}\left[b^{n^2-n+1}u_{k(n-1)}u_{kt}^n + b^{n^2}u_{k(t-1)}^n u_k - b^{n^2-n+1}u_{k(t-1)}u_{kn}u_{kt}^{n-1}\right]$$
(3.2)

Proof. (Induction on n) When $n = 2$, we have

$$\det g_{k,2,t} = \begin{vmatrix} u_{(3+t)} & -bu_{(2+t)} \\ u_{(2+t)} & -bu_{(1+t)} \end{vmatrix} = -b\left(u_{(3+t)}u_{(1+t)} - u_{(2+t)}^2\right) = b^{t+2}u_k^2.$$

Substituting $n = 2$ in the RHS of (3.2), we get

$$\frac{(-1)^{2k}}{u_k}\left[b^3 u_k u_{kt}^2 + b^4 u_{k(t-1)}^2 u_k - b^3 u_{k(t-1)}u_{2k}u_{kt}\right]$$
$$= b^3\left(u_{kt}^2 + bu_{k(t-1)}^2 - u_{k(t-1)}v_k u_{kt}\right)$$
$$= b^3\left(u_{kt}^2 - u_{k(t+1)}u_{k(t-1)}\right) = b^{t+2}u_k^2,$$

as claimed. We assume that the claim is true for $n - 1$. Now we prove that the claim is true for n. By the induction hypothesis and (3.1), we write for odd integer k,

$$\det g_{k,n,t}$$
$$= (-1)^n\, b^{n^2-n+t}u_{k(n-1)}u_k u_{kt}^{n-2} - b^{2n-1}u_{k(t-1)}\frac{(-1)^{k(n-1)}}{u_k}$$
$$\times \left[b^{n^2-3n+3}u_{k(n-2)}u_{kt}^{n-1} + b^{(n-1)^2}u_{k(t-1)}^{n-1}u_k - b^{n^2-3n+3}u_{k(t-1)}u_{k(n-1)}u_{kt}^{n-2}\right]$$
$$= (-1)^{k(n-1)+1}\, b^{n^2}u_{k(t-1)}^n + (-1)^{k(n-1)}\, b^{n^2-n+1}\frac{u_{k(t-1)}u_{kt}^{n-1}u_{kn}}{u_k} +$$
$$+ u_{kt}^{n-2}u_{k(n-1)}\left[(-1)^{kn}\, b^{n^2-n+t}u_k - (-1)^{k(n-1)}\, b^{n^2-n+1}\frac{u_{k(t+1)}u_{k(t-1)}}{u_k}\right]$$
$$= (-1)^{k(n-1)+1}\, b^{n^2}u_{k(t-1)}^n + (-1)^{k(n-1)}\, b^{n^2-n+1}\frac{u_{k(t-1)}u_{kt}^{n-1}u_{kn}}{u_k} +$$
$$+ (-1)^{kn}\, b^{n^2-n+1}\frac{u_{kt}^{n-2}u_{k(n-1)}}{u_k}\left[b^{t-1}u_k^2 + u_{k(t+1)}u_{k(t-1)}\right]$$
$$= \frac{(-1)^{kn}}{u_k}\left[b^{n^2-n+1}u_{k(n-1)}u_{kt}^n + b^{n^2}u_{k(t-1)}^n u_k - b^{n^2-n+1}u_{k(t-1)}u_{kn}u_{kt}^{n-1}\right].$$

Thus the proof is complete. □

Lemma 3.3. *For $n > 1$,*

$$\det h_{k,n,t} = (-1)^{n+1}\, b^{n(n-1)}u_{k(n+t)}u_{kt}^{n-1}.$$

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Proof. Expanding $\det h_{k,n,t}$ with respect to the last row gives us

$\det h_{k,n,t}$

$$
= \begin{vmatrix}
u_{k(2n+t-1)} & b^n u_{kt} & -b^{n+1}u_{k(t-1)} & 0 & \cdots & 0 \\
u_{k(2n+t-2)} & 0 & b^n u_{kt} & -b^{n+1}u_{k(t-1)} & \ddots & \vdots \\
\vdots & \vdots & 0 & b^n u_{kt} & \ddots & 0 \\
\vdots & \vdots & \vdots & 0 & \ddots & -b^{n+1}u_{k(t-1)} \\
u_{k(n+t+1)} & 0 & \vdots & \vdots & \ddots & b^n u_{kt} \\
u_{k(n+t)} & 0 & 0 & \cdots & \cdots & 0
\end{vmatrix}
$$

$$
= u_{k(n+t)} \left(-1\right)^{n+1} \left(b^n u_{kt}\right)^{n-1}
$$

$$
= (-1)^{n+1} b^{n(n-1)} u_{k(n+t)} u_{kt}^{n-1},
$$

as claimed. $\qquad\square$

Lemma 3.4. *For $n > 1$ and $k, t > 0$,*

$$
\begin{aligned}
v_{kn} &= \left(v_k u_{kn} - 2b u_{k(n-1)}\right)/u_k, \\
u_{k(n+t)} &= \left(u_{k(n+1)} u_{kt} - b u_{kn} u_{k(t-1)}\right)/u_k.
\end{aligned}
$$

Proof. The claims are obtained from the Binet formulae of $\{u_n\}$ and $\{v_n\}$. $\qquad\square$

Theorem 3.5. *For $n \geq 2$ and all integer t,*

$$
\det R_{k,n,t} = b^{n(n-1)} \left((-1)^{kn-1} v_{kn} u_{kt}^n + (-1)^{kn} b^n u_{k(t-1)}^n\right), \qquad (3.3)
$$

where k is an odd integer.

Proof. From the definitions of $g_{k,n,t}$ and $h_{k,n,t}$, we see that

$$
\det R_{k,n,t} = \det g_{k,n,t} + \det h_{k,n,t}.
$$

So the proof follows from Lemmas 3.2, 3.3 and 3.4. $\qquad\square$

When $t = n$ in (3.2) and (3.3), we have the following result.

Corollary 3.6. *For $n > 1$,*

$$
\begin{aligned}
\det g_{k,n,n} &= (-1)^{kn} b^{n^2} u_{k(n-1)}^n, \\
\det R_{k,n,n} &= (-1)^{kn} b^{n(n-1)} \left(-v_{kn} u_{kn}^n + b^n u_{k(n-1)}^n\right).
\end{aligned}
$$

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