Identities with squares of binomial coefficients: An elementary and explicit approach

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Abstract. In [**1**], the author presents a recursive method to find closed forms for two kinds of sums involving squares of binomial coefficients. We give an elementary and explicit approach to compute these two kinds of sums. It is based on a triangle of numbers which is akin to the Stirling subset numbers.

1. Introduction

In [**1**], the author presents a recursive method to find closed forms for the sums

$$S_m(n) = \sum_{k=0}^{n-1} k^m \binom{2n}{k}^2, \qquad n \geqslant 1$$

and

$$T_m(n) = \sum_{k=0}^{n-2} k^m \binom{2n-1}{k}^2, \qquad n \geqslant 2$$

for a fixed integer $m \geqslant 0$. The method is based on the relationships between the sums $S_m(n)$ and $T_m(n)$, given by

$$S_m(n) = 4n^2 \sum_{i=0}^{m-2} \binom{m-2}{i} T_i(n), \qquad n \geqslant 1$$

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and

$$T_m(n) = (2n-1)^2 \sum_{i=0}^{m-2} \binom{m-2}{i} \left[S_i(n-1) - (n-2)^i \binom{2n-2}{n}^2 \right], \qquad n \geqslant 2$$

for $m > 1$. Since S_0, S_1, T_0 and T_1 are known, these recursions can be used to compute $S_m(n)$ and $T_m(n)$. It should be noted that these formulae are not explicit but recursive in nature.

In the present paper, we will provide explicit evaluations of these sums. The method is completely elementary. The computations will be reduced to the instance $m = 0$; this is, however, simple, since, by symmetry, the sums are basically half of the full sum which is evaluated by the Vandermonde convolution. This reduction is achieved by replacing the powers k^m by a linear combination of $k^{\underline{h}}k^{\underline{h}}$ and $k^{\underline{h}}k^{\underline{h-1}}$. The coefficients that appear here resemble the Stirling subset numbers (Stirling numbers of the second kind). We recall the notation $x^{\underline{h}} = x(x-1)\dots(x-h+1)$.

In the next section, auxiliary sums will be evaluated using the above mentioned symmetry argument. In the final section, the connecting coefficients will be discussed, leading to the final evaluations.

Note that these evaluations involve roughly $\mathsf{const} \cdot m$ terms. Since we consider m to be a given (small) number, the resulting formulae are in closed form.

2. Auxiliary sums

We evaluate here in an elementary way 4 families of sums.

$$\begin{aligned} A_m(n) &= \sum_{k=0}^{n-1} k^{\underline{m}} k^{\underline{m}} \binom{2n}{k}\binom{2n}{k} \\ &= (2n)^{\underline{m}}(2n)^{\underline{m}} \sum_{k=m}^{n-1} \binom{2n-m}{k-m}\binom{2n-m}{k-m} \\ &= (2n)^{\underline{m}}(2n)^{\underline{m}} \sum_{k=0}^{n-1-m} \binom{2n-m}{k}\binom{2n-m}{k}. \end{aligned}$$

Now we distinguish two cases, first assume that $m = 2h$. Then

$$\begin{aligned} A_{2h}(n) &= (2n)^{\underline{2h}}(2n)^{\underline{2h}} \sum_{k=0}^{n-1-2h} \binom{2n-2h}{k}\binom{2n-2h}{k} \\ &= (2n)^{\underline{2h}}(2n)^{\underline{2h}} \sum_{k=0}^{n-1-h} \binom{2n-2h}{k}\binom{2n-2h}{k} \\ &\quad - (2n)^{\underline{2h}}(2n)^{\underline{2h}} \sum_{k=n-2h}^{n-1-h} \binom{2n-2h}{k}\binom{2n-2h}{k} \\ &= (2n)^{\underline{2h}}(2n)^{\underline{2h}} \frac{1}{2}\left[\binom{4n-4h}{2n-2h} - \binom{2n-2h}{n-h}\binom{2n-2h}{n-h} \right] \end{aligned}$$

$$- (2n)^{\underline{2h}}(2n)^{\underline{2h}} \sum_{k=n-2h}^{n-1-h} \binom{2n-2h}{k}\binom{2n-2h}{k}.$$

If $m = 2h+1$, then

$$\begin{aligned}
A_{2h+1}(n) &= (2n)^{\underline{2h+1}}(2n)^{\underline{2h+1}} \sum_{k=0}^{n-2-2h} \binom{2n-2h-1}{k}\binom{2n-2h-1}{k} \\
&= (2n)^{\underline{2h+1}}(2n)^{\underline{2h+1}} \sum_{k=0}^{n-h-1} \binom{2n-2h-1}{k}\binom{2n-2h-1}{k} \\
&\quad - (2n)^{\underline{2h+1}}(2n)^{\underline{2h+1}} \sum_{k=n-2h-1}^{n-h-1} \binom{2n-2h-1}{k}\binom{2n-2h-1}{k} \\
&= (2n)^{\underline{2h+1}}(2n)^{\underline{2h+1}} \frac{1}{2}\binom{4n-4h-2}{2n-2h-1} \\
&\quad - (2n)^{\underline{2h+1}}(2n)^{\underline{2h+1}} \sum_{k=n-2h-1}^{n-h-1} \binom{2n-2h-1}{k}\binom{2n-2h-1}{k}.
\end{aligned}$$

Now we consider a second class of sums, namely

$$\begin{aligned}
B_m(n) &:= \sum_{k=0}^{n-1} k^{\underline{m}} k^{\underline{m-1}} \binom{2n}{k}\binom{2n}{k} \\
&= (2n)^{\underline{m}}(2n)^{\underline{m-1}} \sum_{k=m}^{n-1} \binom{2n-m}{k-m}\binom{2n-m+1}{k-m+1} \\
&= (2n)^{\underline{m}}(2n)^{\underline{m-1}} \sum_{k=0}^{n-m-1} \binom{2n-m}{k}\binom{2n-m+1}{k+1}.
\end{aligned}$$

Now let $m = 2h$:

$$\begin{aligned}
B_{2h}(n) &= (2n)^{\underline{2h}}(2n)^{\underline{2h-1}} \sum_{k=0}^{n-2h-1} \binom{2n-2h}{k}\binom{2n-2h+1}{k+1} \\
&= (2n)^{\underline{2h}}(2n)^{\underline{2h-1}} \sum_{k=0}^{n-2h-1} \binom{2n-2h}{k}\binom{2n-2h}{k+1} \\
&\quad + (2n)^{\underline{2h}}(2n)^{\underline{2h-1}} \sum_{k=0}^{n-2h-1} \binom{2n-2h}{k}\binom{2n-2h}{k} \\
&= (2n)^{\underline{2h}}(2n)^{\underline{2h-1}} \sum_{k=0}^{n-h-1} \binom{2n-2h}{k}\binom{2n-2h}{k+1} \\
&\quad - (2n)^{\underline{2h}}(2n)^{\underline{2h-1}} \sum_{k=n-2h}^{n-h-1} \binom{2n-2h}{k}\binom{2n-2h}{k+1} + \frac{1}{2n-2h+1} A_{2h}(n)
\end{aligned}$$

$$= (2n)^{\underline{2h}}(2n)^{\underline{2h-1}}\frac{1}{2}\binom{4n-4h}{2n-2h+1}$$
$$- (2n)^{\underline{2h}}(2n)^{\underline{2h-1}}\sum_{k=n-2h}^{n-h-1}\binom{2n-2h}{k}\binom{2n-2h}{k+1} + \frac{1}{2n-2h+1}A_{2h}(n).$$

Now let $m = 2h+1$:

$$\begin{aligned}
B_{2h+1}(n) &= (2n)^{\underline{2h+1}}(2n)^{\underline{2h}}\sum_{k=0}^{n-2h}\binom{2n-2h-1}{k}\binom{2n-2h}{k+1}\\
&= (2n)^{\underline{2h+1}}(2n)^{\underline{2h}}\sum_{k=0}^{n-2h}\binom{2n-2h-1}{k}\binom{2n-2h-1}{k+1}\\
&\quad + (2n)^{\underline{2h+1}}(2n)^{\underline{2h}}\sum_{k=0}^{n-2h}\binom{2n-2h-1}{k}\binom{2n-2h-1}{k}\\
&= (2n)^{\underline{2h+1}}(2n)^{\underline{2h}}\sum_{k=0}^{n-h-1}\binom{2n-2h-1}{k}\binom{2n-2h-1}{k+1}\\
&\quad - (2n)^{\underline{2h+1}}(2n)^{\underline{2h}}\sum_{k=n-2h+1}^{n-h-1}\binom{2n-2h-1}{k}\binom{2n-2h-1}{k+1}\\
&\quad + (2n)^{\underline{2h+1}}(2n)^{\underline{2h}}\sum_{k=0}^{n-2-2h}\binom{2n-2h-1}{k}\binom{2n-2h-1}{k}\\
&\quad + (2n)^{\underline{2h+1}}(2n)^{\underline{2h}}\sum_{k=n-1-2h}^{n-2h}\binom{2n-2h-1}{k}\binom{2n-2h-1}{k}\\
&= (2n)^{\underline{2h+1}}(2n)^{\underline{2h}}\frac{1}{2}\left[\binom{4n-4h-2}{2n-2h}+\binom{2n-2h-1}{n-h}\binom{2n-2h-1}{n-h-1}\right]\\
&\quad - (2n)^{\underline{2h+1}}(2n)^{\underline{2h}}\sum_{k=n-2h+1}^{n-h-1}\binom{2n-2h-1}{k}\binom{2n-2h-1}{k+1}\\
&\quad + \frac{1}{2n-2h}A_{2h+1}(n)\\
&\quad + (2n)^{\underline{2h+1}}(2n)^{\underline{2h}}\left(\binom{2n-2h-1}{n-2h}^2+\binom{2n-2h-1}{n-2h-1}^2\right).
\end{aligned}$$

We need two more sums which can be reduced to the previous ones.

$$\begin{aligned}
C_m(n) &= \sum_{k=0}^{n-2}k^{\underline{m}}k^{\underline{m}}\binom{2n-1}{k}\binom{2n-1}{k}\\
&= (2n-1)^{\underline{m}}(2n-1)^{\underline{m}}\sum_{k=m}^{n-2}\binom{2n-1-m}{k-m}\binom{2n-1-m}{k-m}
\end{aligned}$$

$$= (2n-1)^{\underline{m}}(2n-1)^{\underline{m}} \sum_{k=0}^{n-2-m} \binom{2n-1-m}{k}\binom{2n-1-m}{k}$$
$$= \frac{1}{4n^2} A_{m+1}(n).$$

$$D_m(n) = \sum_{k=0}^{n-2} k^{\underline{m}} k^{\underline{m-1}} \binom{2n-1}{k}\binom{2n-1}{k}$$
$$= (2n-1)^{\underline{m}}(2n-1)^{\underline{m-1}} \sum_{k=m}^{n-2} \binom{2n-1-m}{k-m}\binom{2n-m}{k+1-m}$$
$$= (2n-1)^{\underline{m}}(2n-1)^{\underline{m-1}} \sum_{k=0}^{n-2-m} \binom{2n-1-m}{k}\binom{2n-m}{k+1}$$
$$= \frac{1}{4n^2} B_{m+1}(n).$$

3. Evaluation of Slavik's sums

First, we consider

$$S_m(n) = \sum_{k=0}^{n-1} k^m \binom{2n}{k}\binom{2n}{k}.$$

In order to do so, we write k^m as a linear combination of $k^{\underline{h}}k^{\underline{h}}$ and $k^{\underline{h}}k^{\underline{h-1}}$. It is clear that this can be done in a unique way, since the polynomials $x^{\underline{h}}x^{\underline{h}}$ and $x^{\underline{h}}x^{\underline{h-1}}$ form a basis for the vector space of the polynomials over $\mathbb{R}$.

Then we are left with the sums

$$A_h(n) = \sum_{k=0}^{n-1} k^{\underline{h}}k^{\underline{h}} \binom{2n}{k}\binom{2n}{k} \quad \text{and} \quad B_h(n) = \sum_{k=0}^{n-1} k^{\underline{h}}k^{\underline{h-1}} \binom{2n}{k}\binom{2n}{k}$$

which have been evaluated in the previous section.

For the other family $T_m(n)$, the approach is similar:

$$T_m(n) = \sum_{k=0}^{n-2} k^m \binom{2n-1}{k}\binom{2n-1}{k}$$

is reduced to a linear combination of

$$C_h(n) = \sum_{k=0}^{n-2} k^{\underline{h}}k^{\underline{h}} \binom{2n-1}{k}\binom{2n-1}{k}$$

and

$$D_h(n) = \sum_{k=0}^{n-2} k^{\underline{h}}k^{\underline{h-1}} \binom{2n-1}{k}\binom{2n-1}{k},$$

which are already evaluated.

The only thing that is left is to identify the coefficients in the above mentioned linear combinations.

Denote

$$p_k(x) = \begin{cases} x^{\underline{k}} x^{\underline{k}} & \text{if } k \text{ is even,} \\ x^{\underline{k}} x^{\underline{k-1}} & \text{if } k \text{ is odd.} \end{cases}$$

For convenience, we state the following simple result as a proposition.

PROPOSITION 3.1. *For nonnegative integers n,*

$$x^n = \sum_{k=0}^{n} a_{n,k} p_k(x),$$

where

$$a_{n+1,k} = a_{n,k-1} + \lfloor k/2 \rfloor a_{n,k}$$

with initial conditions $a_{n,0} = [n = 0]$ and $a_{n,n} = 1$.

PROOF. Since

$$x p_{2k}(x) = p_{2k+1}(x) + k p_{2k}(x)$$

and

$$x p_{2k+1}(x) = p_{2k+2}(x) + k p_{2k+1}(x),$$

we write

$$x^{n+1} = \sum_{k=0}^{n+1} a_{n+1,k} p_k(x) = \sum_{k=0}^{n} a_{n,k} x p_k(x)$$
$$= \sum_{k=0}^{n} a_{n,k} \Big[p_{k+1}(x) + \Big\lfloor \frac{k}{2} \Big\rfloor p_k(x) \Big].$$

Now comparing coefficients, we get the recurrence

$$a_{n+1,k} = a_{n,k-1} + \Big\lfloor \frac{k}{2} \Big\rfloor a_{n,k}.$$

Plugging in $x = 0$ leads to $a_{n,0} = [n = 0]$; the comparison of the highest power n leads to $a_{n,n} = 1$, as claimed. □

For interest, here is a small table of these values.

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	0	1					
2	0	0	1				
3	0	0	1	1			
4	0	0	1	2	1		
5	0	0	1	3	4	1	
6	0	0	1	4	11	6	1

References

[1] A. Slavik, Identities with squares of binomial coefficients, Ars Combinatoria, 113 (2014), 377–383.

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