Noname manuscript No.
(will be inserted by the editor)

Quadratic sums of Gaussian q-binomial coefficients and Fibonomial Coefficients

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Received: date / Accepted: date

Abstract In this paper, we will evaluate quadratic sums of Gaussian q-binomial coefficients with two additional parameters. A general summation theorem will be proved by a combination of the Heine transformation, the q-Pfaff-Saalschutz theorem and the q-Kummer sum. As consequences, several identities will be presented for the generalized Fibonomial-Lucanomial coefficients by specifying the parameter p and the base q to particular values.

Keywords Basic hypergeometric series · q-binomial coefficient · Heine transformation · q-Pfaff-Saalschutz summation formula · q-Kummer sum · Fibonomial and Lucanomial coefficients

1 Introduction

Denote by $\mathbb{Z}$ and $\mathbb{N}$, respectively, the sets of integers and natural numbers with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For two indeterminates p and q, the q-Pochhammer symbol reads as $(p;q)_0 = 1$ and $(p;q)_n = (1-p)(1-pq)\cdots(1-pq^{n-1})$ for $n \in \mathbb{N}$. Then for

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$n,\ k \in \mathbb{N}_0$, the generalized Gaussian binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{(p;q)_n}{(p;q)_k (p;q)_{n-k}} \text{ with } \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = 0 \text{ for } k < 0 \text{ or } k > n$$

which will become, when $p = q$, the usual q-binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

Define the two Fibonacci–like sequences $\{U_n\}$ and $\{V_n\}$ by the second order linear recurrence relations

$$U_n = pU_{n-1} + U_{n-2} \quad \text{and} \quad V_n = pV_{n-1} + V_{n-2}$$

with the initial conditions $U_0 = 0,\ U_1 = 1$, and, $V_0 = 2,\ V_1 = p$, respectively. Their Binet forms are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n (1 + q^n)$$

where $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$ with $\mathbf{i} = \sqrt{-1}$ being the imaginary unit. When $\alpha = \frac{1+\sqrt{5}}{2}$ (or equivalently $q = \frac{1-\sqrt{5}}{1+\sqrt{5}}$), the sequences $\{U_n\}$ and $\{V_n\}$ will reduce, respectively, to the Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$. Instead, for $\alpha = 1 + \sqrt{2}$ (or equivalently $q = \frac{1-\sqrt{2}}{1+\sqrt{2}}$), these sequences $\{U_n\}$ and $\{V_n\}$ will become the Pell sequence $\{P_n\}$ and the Pell–Lucas sequence $\{Q_n\}$.

For $m, n, k \in \mathbb{N}$ with $n \geq k \geq 1$, define the generalized Fibonomial coefficients with indices being multiples of m by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U:m} = \frac{U_m U_{2m} \dots U_{nm}}{(U_m U_{2m} \dots U_{km}) \left(U_m U_{2m} \dots U_{m(n-k)} \right)} = \prod_{j=1}^{k} \frac{U_{m(n-j+1)}}{U_{mj}}$$

as well as the boundary conditions $\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\}_{U:m} = \left\{ \begin{matrix} n \\ n \end{matrix} \right\}_{U:m} = 1$. For more details about the Fibonomial coefficients and their properties, the interested reader may refer to [3, 4, 11, 12].

There exist several variants of Fibonomial coefficients. We record them briefly for the subsequent applications.

– When $m = 1$, the above generalized Fibonomial coefficients reduce to the usual ones, denoted shortly by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U} = \frac{U_n U_{n-1} \dots U_{n-k+1}}{U_1 U_2 \dots U_k}.$$

– When $U_n = F_n$, the last coefficients reduce further to the Fibonomial coefficients:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{F} = \frac{F_n F_{n-1} \dots F_{n-k+1}}{F_1 F_2 \dots F_k}.$$

– Similarly, when $U_n = P_n$, we get the following Pellnomial coefficients:

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{P} = \frac{P_n P_{n-1} \dots P_{n-k+1}}{P_1 P_2 \dots P_k}.$$

– Furthermore, we will use the *generalized Lucanomial coefficients*

$$\left\langle {n \atop k} \right\rangle_V = \frac{V_n V_{n-1} \dots V_{n-k+1}}{V_1 V_2 \dots V_k}.$$

– When $V_n = L_n$, the last coefficients reduce to the usual Lucanomial coefficients:

$$\left\langle {n \atop k} \right\rangle_L = \frac{L_n L_{n-1} \dots L_{n-k+1}}{L_1 L_2 \dots L_k}.$$

– When $V_n = Q_n$, we get the Pell-Lucanomial coefficients:

$$\left\langle {n \atop k} \right\rangle_Q = \frac{Q_n Q_{n-1} \dots Q_{n-k+1}}{Q_1 Q_2 \dots Q_k}.$$

Our approach will essentially be based on the following connection between the generalized Fibonomial and Gaussian q-binomial coefficients

$$\left\{ {n \atop k} \right\}_{U:m} = \alpha^{mk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^m} \quad \text{with} \quad q = -\alpha^{-2}.$$

An analogous relation holds for the generalized Lucanomial coefficients:

$$\left\langle {n \atop k} \right\rangle_V = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{-q} \quad \text{with} \quad q = -\alpha^{-2}.$$

There are various sums, in the literature, involving Gaussian q-binomial coefficients with certain weight functions. Quite recently, a number of sums containing generalized Fibonomial coefficients are evaluated, in closed forms, in [4, 7, 8, 11, 12]. Observing that there exists a correspondence between these two classes of finite sums via $q = \beta/\alpha$, we can examine and evaluate one class of sums from another class according to the appropriate convenience. Hence it is possible to categorize the sums on the generalized Fibonomial coefficients in the current literature according to the number of Gaussian coefficients (or Fibonomial coefficients) appearing in these q-binomial sums. Such attempts have been made in [5,6] for the sums containing only one q-binomial coefficient and in [9, 10] for the sums involving two coefficients.

Now we will focus on the sums of the second kind. Much recently, Kılıç and Prodinger [7] give a systematic treatment to certain sums of squares of Fibonomial coefficients with finite products of generalized Fibonacci and Lucas numbers as coefficients. The technique consists of rewriting everything in terms of another variable q and then making use of generating functions and Rothe's identity from classical q-calculus.

For example, in order to evaluate in closed form

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\}_U^2 U_{\lambda_1 k+\nu_1} \dots U_{\lambda_s k+\nu_s}$$

where λ and ν_i are integers with $\lambda_i \geq 1$, the authors in [7] translated this into the following q-binomial sum:

$$(1-q)^{-s}\sum_{k=0}^{n}(-1)^{k(n-1)}q^{-k(n-k)}\begin{bmatrix}n\\k\end{bmatrix}_q^2 \mathbf{i}^{(\lambda_1+\cdots+\lambda_s)k+\nu_1+\cdots+\nu_s-s}$$
$$\times q^{\frac{s}{2}-\frac{k}{2}(\lambda_1+\cdots+\lambda_s)-\frac{1}{2}(\nu_1+\cdots+\nu_s)}(1-q^{\lambda_1 k+\nu_1})\cdots(1-q^{\lambda_s k+\nu_s}),$$

which results in a linear combination of square sums of the Gaussian binomial coefficients:

$$\sum_{k=0}^{n}(-1)^k q^{k^2+\mu k-nk}\begin{bmatrix}n\\k\end{bmatrix}_q^2 \quad \text{where} \quad \mu\in\mathbb{Z}.$$

Their results can be highlighted by the following formula

$$\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\k\end{bmatrix}_q^2(-1)^k q^{k^2-2kn-3k}(1-2q^{2k}+q^{4k})$$
$$=2(-1)^{n+1}q^{-n^2-2n-2}\frac{(1+q)(1-q^{2n+1})(1-q^{2n+2})}{(1+q^{2n})}\begin{bmatrix}2n+1\\n\end{bmatrix}_{q^2}.$$

Furthermore, Kılıç and Prodinger [8] investigated sums containing the square of Fibonomial coefficients with the powers of generalized Fibonacci and Lucas numbers, over half the natural summation range. They discussed both, the Fibonacci and Lucas instances, where the range of summations is over all non-negative integers (i. e., about half of the possible number of terms); and treated, systematically in closed forms for $\lambda,\mu,\nu\in\mathbb{Z}$, the following two kinds of sums

$$\sum_{k=0}^{n}\begin{Bmatrix}2n\\n-k\end{Bmatrix}_U^2(-1)^{k\lambda}V_{k\nu}^{\mu} \qquad \text{and} \qquad \sum_{k=0}^{n}\begin{Bmatrix}2n\\n-k\end{Bmatrix}_U^2(-1)^{k\lambda}U_{k\nu}^{\mu};$$
$$\sum_{k=-n}^{n}\begin{Bmatrix}2n\\n-k\end{Bmatrix}_U^2(-1)^{k\lambda}V_{k\nu}^{\mu} \qquad \text{and} \qquad \sum_{k=-n}^{n}\begin{Bmatrix}2n\\n-k\end{Bmatrix}_U^2(-1)^{k\lambda}U_{k\nu}^{\mu}.$$

A typical formula can be reproduced as follows:

$$\sum_{k=0}^{n}\begin{bmatrix}2n\\n+k\end{bmatrix}_q^2(-1)^k q^{k(k-2)}\left(1-q^{2k}\right)^2$$
$$=-q^{-1}(1+q)\left(1-q^{2n-1}\right)\left(1-q^{2n}\right)\begin{bmatrix}2n-2\\n-1\end{bmatrix}_{q^2}.$$

In this paper, we shall investigate the following quadratic sums of Gaussian q-binomial coefficients with two additional parameters

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}2m+n\\m+k\end{bmatrix}_{p,q}q^{k^2-nk+k\lambda}$$

for a real number p and $m, n, \lambda \in \mathbb{N}_0$. The next section will constitute the central part of the paper, where the main theorem together with four useful corollaries will be established. This will be fulfilled by an intrinsic combination of the Heine transformation, the q-Pfaff-Saalschutz theorem and the q-Kummer sum. Then they will be specified in Section 3 to eight q-binomial identities. As consequences, several identities on generalized Fibonomial-Lucanomial coefficients will be deduced finally in Section 4, which not only contain new formulae, but also cover various known results on square sums of the Gaussian q-binomial coefficients when the parameters are specialized. For instance, when $p = q$ and $m = 0$, our results will recover the formulae of [7, 8] in accordance with different values of λ.

2 Evaluation of q-Binomial Sums

In order to shorten the lengthy expressions, we shall write $a \equiv_b c$ for that "a is congruent to c modulo b". Then for $m, n \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$, we are going to evaluate the following q-binomial sums:

$$\Omega(m, n; \lambda) := \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2m+n \\ m+k \end{bmatrix}_{p,q} q^{k^2 - nk + k\lambda}. \tag{2.1}$$

By reversing the summation order $k \to n - k$, we get the reciprocal relation:

$$\Omega(m, n; \lambda) = (-1)^n q^{n\lambda} \times \Omega(m, n; -\lambda). \tag{2.2}$$

Therefore, it suffices to work out closed formula for the case $\lambda \in \mathbb{N}_0$.

Theorem 1 ($m, n, \lambda \in \mathbb{N}_0$)

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2m+n \\ m+k \end{bmatrix}_{p,q} q^{k^2 - nk + k\lambda} = q^{n\lambda} \begin{bmatrix} 2m+n \\ m \end{bmatrix}_{p,q} \frac{(-p; q)_{m+n-\lambda}}{(-p; q)_{m-\lambda}}$$
$$\times \sum_{\substack{i=0 \\ i \equiv_2 n}}^{\lambda} q^i \frac{(q^{-\lambda}; q)_i (q^{-n}; q)_i (p^2 q^{2m+n-1}; q)_i}{(q; q)_i (-pq^{m-\lambda}; q)_i} \frac{(q; q^2)_{\frac{n-i}{2}}}{(p^2 q^{2m}; q^2)_{\frac{n+i}{2}}} (-q)^{-\frac{(n-i)^2}{4}}.$$

This formula is remarkable in the sense that it contains only a few terms when λ is a small integer. Therefore, it is quite efficient for computation of the Ω-sum in this case.

Proof. Writing the q-binomial coefficients in terms of shifted factorials

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = (-1)^k \frac{(q^{-n}; q)_k}{(q; q)_k} q^{nk - \binom{k}{2}},$$
$$\begin{bmatrix} 2m+n \\ m+k \end{bmatrix}_{p,q} = (-p)^k \begin{bmatrix} 2m+n \\ m \end{bmatrix}_{p,q} \frac{(q^{1-m-n}/p; q)_k}{(pq^m; q)_k} q^{mk + nk - \binom{k+1}{2}};$$

we first transform the Ω-sum into the following q-series

$$\Omega(m,n;\lambda) = \begin{bmatrix} 2m+n \\ m \end{bmatrix}_{p,q} {}_2\phi_1 \left[\begin{matrix} q^{-n},\ q^{1-m-n}/p \\ pq^m \end{matrix}\middle| q; -pq^{m+n+\lambda}\right]$$

where we have adopted the q-series notation from Chen–Chu [1] and the text book by Gasper and Rahman [2]. Now applying the Heine transformation (cf. [2, III-2])

$${}_2\phi_1 \left[\begin{matrix} a,\ b \\ c \end{matrix}\middle| q; z\right] = \frac{(bz;q)_\infty (c/b;q)_\infty}{(c;q)_\infty (z;q)_\infty} {}_2\phi_1 \left[\begin{matrix} abz/c,\ b \\ bz \end{matrix}\middle| q; c/b\right]$$

we can further reformulate the Ω-sum as

$$\begin{aligned} \Omega(m,n;\lambda) &= \begin{bmatrix} 2m+n \\ m \end{bmatrix}_p {}_2\phi_1 \left[\begin{matrix} q^{-n}, q^{1-m-n}/p \\ pq^m \end{matrix}\middle| q; -pq^{m+n+\lambda}\right] \\ = \frac{(-pq^{m+\lambda};q)_n}{(pq^m;q)_n} &\begin{bmatrix} 2m+n \\ m \end{bmatrix}_p {}_2\phi_1 \left[\begin{matrix} q^{-n}, -q^{1+\lambda-m-n}/p \\ -pq^{\lambda+m} \end{matrix}\middle| q; pq^{m+n}\right]. \end{aligned} \tag{2.3}$$

Recalling the q-Saalschutz summation formula (cf. [2, II-12])

$$\frac{(c/a;q)_\lambda (c/b;q)_\lambda}{(c;q)_\lambda (c/ab;q)_\lambda} = {}_3\phi_2 \left[\begin{matrix} q^{-\lambda},\ a, & b \\ c, & q^{1-\lambda}ab/c \end{matrix}\middle| q; q\right]$$

we have the following equality

$$\frac{(-q^{1+k-m-n}/p;q)_\lambda (-pq^{m+n};q)_\lambda}{(-pq^{k+m};q)_\lambda (-q^{1-m}/p;q)_\lambda} = {}_3\phi_2 \left[\begin{matrix} q^{-\lambda}, & q^{k-n}, & p^2q^{2m+n-1} \\ & -pq^{m+k}, & -pq^{m-\lambda} \end{matrix}\middle| q; q\right].$$

By inserting this relation inside the ultimate ${}_2\phi_1$-series expression for the Ω-sum and then making use of the following relations

$$\begin{aligned} &(q^{-n};q)_k (q^{k-n};q)_i = (q^{-n};q)_i (q^{i-n};q)_k, \\ &\frac{(-q^{1+\lambda-m-n}/p;q)_k}{(-q^{1+k-m-n}/p;q)_\lambda} = \frac{(-q^{1-m-n}/p;q)_k}{(-q^{1-m-n}/p;q)_\lambda}, \\ &\frac{(-pq^{m+k};q)_\lambda}{(-pq^{m+k};q)_i} = \frac{(-pq^m;q)_\lambda (-pq^{m+\lambda};q)_k}{(-pq^m;q)_i (-pq^{m+i};q)_k}; \end{aligned}$$

we can manipulate the double sum below

$$\begin{aligned} &{}_2\phi_1 \left[\begin{matrix} q^{-n},\ -q^{1+\lambda-m-n}/p \\ -pq^{\lambda+m} \end{matrix}\middle| q; pq^{m+n}\right] \\ &= \sum_{k=0}^{n} \frac{(q^{-n};q)_k (-q^{1+\lambda-m-n}/p;q)_k}{(q;q)_k (-pq^{\lambda+m};q)_k} p^k q^{mk+nk} \\ &\times \frac{(-pq^{k+m};q)_\lambda (-q^{1-m}/p;q)_\lambda}{(-q^{1+k-m-n}/p;q)_\lambda (-pq^{m+n};q)_\lambda} \end{aligned}$$

$$\times \sum_{i=0}^{\lambda} q^i \frac{(q^{-\lambda};q)_i (q^{k-n};q)_i (p^2q^{2m+n-1};q)_i}{(q;q)_i(-pq^{m+k};q)_i(-pq^{m-\lambda};q)_i}$$
$$= \frac{(-pq^m;q)_\lambda(-q^{1-m}/p;q)_\lambda}{(-pq^{m+n};q)_\lambda(-q^{1-m-n}/p;q)_\lambda}$$
$$\times \sum_{i=0}^{\lambda} q^i \frac{(q^{-\lambda};q)_i (q^{-n};q)_i (p^2q^{2m+n-1};q)_i}{(q;q)_i(-pq^{m};q)_i(-pq^{m-\lambda};q)_i}$$
$$\times \sum_{k=0}^{n} \frac{(q^{i-n};q)_k(-q^{1-m-n}/p;q)_k}{(q;q)_k(-pq^{m+i};q)_k} p^k q^{mk+nk}.$$

The last sum with respect to i can be evaluated, in closed form, as

$$ {}_2\phi_1 \left[\begin{matrix} q^{i-n},\ -q^{1-m-n}/p \\ -pq^{m+i} \end{matrix} \middle| q; pq^{m+n}\right]$$
$$= \frac{(q^2;q^2)_\infty(q^{1+i-n};q^2)_\infty(p^2q^{i+2m+n};q^2)_\infty}{(q;q)_\infty(pq^{m+n};q)_\infty(-pq^{m+i};q)_\infty}$$

where we have employed q-Kummer sum (cf. [2, II-9]):

$$ {}_2\phi_1 \left[\begin{matrix} a,\ c \\ qa/c \end{matrix} \middle| q; -q/c\right] = \frac{(q^2;q^2)_\infty(qa;q^2)_\infty(q^2a/c^2;q^2)_\infty}{(q;q)_\infty(qa/c;q)_\infty(-q/c;q)_\infty}.$$

Observe that $(q^{1+i-n};q^2)_\infty = 0$ for $i < n$ with i and n having the opposite parity. We can further simplify, when i and n have the same parity, the quotient expression:

$$\frac{(q^2;q^2)_\infty(q^{1+i-n};q^2)_\infty(p^2q^{i+2m+n};q^2)_\infty}{(q;q)_\infty(pq^{m+n};q)_\infty(-pq^{m+i};q)_\infty}$$
$$= \frac{(q^{1+i-n};q^2)_\infty}{(q;q^2)_\infty} \frac{(p;q)_{m+n}(-p;q)_{m+i}}{(p^2;q^2)_{m+\frac{n+i}{2}}}$$
$$= \frac{(q^{1+i-n};q^2)_{\frac{n-i}{2}}}{} \frac{(p;q)_{m+n}(-p;q)_{m+i}}{(p^2;q^2)_{m+\frac{n+i}{2}}}$$
$$= (-q)^{-\frac{(n-i)^2}{4}} \frac{(p;q)_{m+n}(-p;q)_{m+i}}{(p^2;q^2)_{m+\frac{n+i}{2}}} (q;q^2)_{\frac{n-i}{2}}.$$

Therefore, we have established the following expression

$$ {}_2\phi_1 \left[\begin{matrix} q^{-n}, -q^{1+\lambda-m-n}/p \\ -pq^{\lambda+m} \end{matrix} \middle| q; pq^{m+n}\right] = \frac{(p;q)_{m+n}(-p;q)_{m+\lambda}(-pq^{m-\lambda};q)_\lambda}{(p^2;q^2)_m(-pq^{m+n};q)_\lambda(-pq^{m+n-\lambda};q)_\lambda}$$
$$\times q^{n\lambda} \sum_{\substack{i=0 \\ i\equiv_2 n}}^{\lambda} q^i \frac{(q^{-\lambda};q)_i(q^{-n};q)_i(p^2q^{2m+n-1};q)_i}{(q;q)_i(-pq^{m-\lambda};q)_i} \frac{(q;q^2)_{\frac{n-i}{2}}}{(p^2q^{2m};q^2)_{\frac{n+i}{2}}} (-q)^{-\frac{(n-i)^2}{4}}.$$

Substituting this into (2.3) and then simplifying the quotient expression

$$\frac{(p;q)_{m+n}(-pq^{m+\lambda};q)_n(-p;q)_{m+\lambda}(-pq^{m-\lambda};q)_\lambda}{(p^2;q^2)_m(pq^m;q)_n(-pq^{m+n};q)_\lambda(-pq^{m+n-\lambda};q)_\lambda}=\frac{(-p;q)_{m+n-\lambda}}{(-p;q)_{m-\lambda}}$$

we confirm the formula stated in Theorem 1. □

By specifying λ to small integers in Theorem 1, we can derive the following interesting summation formulae, where for a real number x, we denote by $\lfloor x\rfloor$ the greatest integer non exceeding x.

Corollary 2 ($\lambda = 0$ in Theorem 1: $m, n, \lambda \in \mathbb{N}_0$)

$$\begin{aligned}\Omega(m,n;0) &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2m+n \\ m+k \end{bmatrix}_{p,q} q^{k^2-nk} \\ &= \frac{(q;q^2)_{\lfloor \frac{n}{2}\rfloor}}{(p^2q^{2m};q^2)_{\lfloor \frac{n}{2}\rfloor}} \begin{bmatrix} 2m+n \\ m \end{bmatrix}_{p,q} \frac{(-p;q)_{m+n}}{(-p;q)_m} \\ &\quad\times \begin{cases} (-q)^{-\frac{n^2}{4}}, & n \equiv_2 0; \\ 0, & n \equiv_2 1. \end{cases}\end{aligned}$$

Corollary 3 ($\lambda = 1$ in Theorem 1: $m, n, \lambda \in \mathbb{N}_0$)

$$\begin{aligned}\Omega(m,n;1) &= \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2m+n \\ m+k \end{bmatrix}_{p,q} q^{k^2-nk+k} \\ &= \frac{q^n(q;q^2)_{\lfloor \frac{n}{2}\rfloor}}{(p^2q^{2m};q^2)_{\lfloor \frac{n}{2}\rfloor}} \begin{bmatrix} 2m+n \\ m \end{bmatrix}_{p,q} \frac{(-p;q)_{m+n-1}}{(-p;q)_{m-1}} \\ &\quad\times \begin{cases} (-q)^{-\frac{n^2}{4}}, & n \equiv_2 0; \\ (-q)^{\frac{(n-1)^2}{-4}} \dfrac{q^{-n}(1-q^n)}{(1-q^{n-2})(1+pq^{m-1})}, & n \equiv_2 1. \end{cases}\end{aligned}$$

Corollary 4 ($\lambda = 2$ in Theorem 1: $m, n, \lambda \in \mathbb{N}_0$)

$$\begin{aligned}&\Omega(m,n;2) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2m+n \\ m+k \end{bmatrix}_{p,q} q^{k^2-nk+2k} \\ &= \frac{q^{2n}(q;q^2)_{\lfloor \frac{n}{2}\rfloor}}{(p^2q^{2m};q^2)_{\lfloor \frac{n}{2}\rfloor}} \begin{bmatrix} 2m+n \\ m \end{bmatrix}_{p,q} \frac{(-p;q)_{m+n-2}}{(-p;q)_{m-2}} \\ &\times \begin{cases} (-q)^{-\frac{n^2}{4}} \left\{1+q^{-1}(-1)^n \dfrac{(1-q^{-n})(1-p^2q^{2m+n-1})}{(1+pq^{m-1})(1+pq^{m-2})}\right\}, & n \equiv_2 0; \\ (-q)^{-\frac{(n-1)^2}{4}} \dfrac{q^{-1-n}(1+q)(1-q^n)}{(1-q^{n-2})(1+pq^{m-2})}, & n \equiv_2 1. \end{cases}\end{aligned}$$

Corollary 5 ($\lambda = 3$ in Theorem 1: $m, n, \lambda \in \mathbb{N}_0$)

$$\Omega(m,n;3) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} 2m+n \\ m+k \end{bmatrix}_{p,q} q^{k^2-nk+3k}$$

$$= \frac{q^{3n}(q;q^2)_{\lfloor \frac{n}{2} \rfloor}}{(p^2q^{2m};q^2)_{\lfloor \frac{n}{2} \rfloor}} \begin{bmatrix} 2m+n \\ m \end{bmatrix}_{p,q} \frac{(-p;q)_{m+n-3}}{(-p;q)_{m-3}}$$

$$\times \begin{cases} (-q)^{-\frac{n^2}{4}} \left\{ 1 + q^{-4}(-1)^n \dfrac{(1-q^3)(1-q^{-n})(q-q^{2m+n}p^2)}{(1-q)(1+pq^{m-2})(1+pq^{m-3})} \right\}, & n \equiv_2 0; \\ (-q)^{-\frac{(n-1)^2}{4}} \dfrac{q^{-2-n}(1-q^3)(1-q^n)}{(1-q)(1-q^{n-2})(1+pq^{m-3})} \times & \\ \left\{ 1 + (-1)^n q^{-n} \dfrac{(1-q)(1-q^{n-1})(1-q^{n-2})(1-p^2q^{2m+n})}{(1-q^3)(1-q^{n-4})(1+pq^{m-1})(1+pq^{m-2})} \right\}, & n \equiv_2 1. \end{cases}$$

3 Eight summation formulae of the Gaussian q-binomials

According to the parity of n, we shall derive five summation formulae by specifying $p = \pm q$ in Corollaries established in the last section. In view of the reciprocal relation (2.2), further three q-binomial formulae will also presented.

Example 1 ($n \to 2n$ in Corollary 2: $p = \pm q$)

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n+2m \\ k+m \end{bmatrix}_{p,q} q^{k^2-2kn}$$

$$= (-1)^n \, q^{-n^2} \frac{(q;q)_{2n}}{(p;q)_{2n}} \begin{bmatrix} 2m+2n \\ m \end{bmatrix}_{p,q} \begin{bmatrix} 2n+m \\ m \end{bmatrix}_{p,q}^{-1} \begin{bmatrix} 2n+m \\ n \end{bmatrix}_{q^2}.$$

Example 2 ($n \to 2n+1$ in Corollary 2: $p = \pm q$)

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n+1+2m \\ k+m \end{bmatrix}_{p,q} q^{k^2-2nk-k} = 0.$$

Example 3 ($n \to 2n$ in Corollary 3: $p = \pm q$)

$$\sum_k (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n+2m \\ k+m \end{bmatrix}_{p,q} q^{k^2-2kn+k}$$

$$= (-q)^{2n-n^2} \frac{(q;q)_{2n}}{(p;q)_{2n}} \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2} \begin{bmatrix} 2n+m-1 \\ m-1 \end{bmatrix}_{-p,q} \begin{bmatrix} 2m+2n \\ m \end{bmatrix}_{p,q} \begin{bmatrix} n+m \\ n \end{bmatrix}_{q^2}^{-1}.$$

Example 4 ($n \to 2n+1$ in Corollary 3: $p = \pm q$)

$$\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n+2m+1 \\ k+m \end{bmatrix}_{p,q} q^{k^2-2nk}$$

$$= (-1)^n \, q^{-n^2} \frac{(q;q)_{2n}}{(p;q)_{2n}} \frac{\left(1-pq^{2n+m+1}\right)\left(1-q^{2n+2m+2}\right)}{\left(1-pq^{2n+2m+1}\right)\left(1+q^{2n+2}\right)\left(1+q^{2n+1}\right)}$$

$$\times \begin{bmatrix} 2n+2 \\ n+1 \end{bmatrix}_{q^2} \begin{bmatrix} 2n+m \\ m \end{bmatrix}_{-p,q} \begin{bmatrix} 2m+2n+2 \\ m \end{bmatrix}_{p,q} \begin{bmatrix} n+m+1 \\ m \end{bmatrix}_{q^2}^{-1}.$$

Example 5 ($n \to 2n+1$ in Corollary 4: $p = \pm q$)

$$\begin{aligned}
&\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n+1+2m \\ k+m \end{bmatrix}_{p,q} q^{k^2-2nk+k} \\
&= (-1)^n\, q^{-n(n-2)} \left(1+pq^{m-1}\right) \frac{(-p;q)_{2n-1}}{(-q;q)_{2n-1}} \\
&\times \frac{(1+q)\left(1-q^{2n}\right)\left(1-q^{2n+1}\right)\left(1-q^{2n+2}\right)\left(1-q^{2n+4}\right)}{(1-q^{2n+2m-4})\,(1-q^{2n+2m-2})\,(1-q^{2n+2m})} \\
&\times \begin{bmatrix} 2m+2n+1 \\ m \end{bmatrix}_{p,q} \begin{bmatrix} m+2n-1 \\ m \end{bmatrix}_{p,q}^{-1} \begin{bmatrix} 2n+m-1 \\ n+2 \end{bmatrix}_{q^2}.
\end{aligned}$$

Moreover, by combining the reciprocal relation (2.2) with Corollaries 3 and 4, we can derive the following three examples.

Example 6 ($n \to 2n$ in Corollary 3: $p = \pm q$)

$$\begin{aligned}
&\sum_k (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n+2m \\ k+m \end{bmatrix}_{p,q} q^{k^2-2kn-k} \\
&= (-1)^n\, q^{-n^2} \frac{(q;q)_{2n}}{(p;q)_{2n}} \begin{bmatrix} 2n \\ n \end{bmatrix}_{q^2} \begin{bmatrix} 2n+m-1 \\ m-1 \end{bmatrix}_{-p,q} \begin{bmatrix} 2m+2n \\ m \end{bmatrix}_{p,q} \begin{bmatrix} m+n \\ m \end{bmatrix}_{q^2}^{-1}.
\end{aligned}$$

Example 7 ($n \to 2n+1$ in Corollary 3: $p = \pm q$)

$$\begin{aligned}
&\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n+1+2m \\ k+m \end{bmatrix}_{p,q} q^{k^2-2nk-2k} \\
&= \frac{(q;q)_{2n+2}}{(p;q)_{2n+2}} \frac{\left(1-pq^{2n+m+1}\right)}{(1+pq^{m-1})\,(1+pq^{m+2n})} \frac{\left(1-q^{2n+2m+2}\right)}{(1-pq^{2n+2m+1})} (-1)^{n+1}\, q^{-(n+1)^2} \\
&\times \begin{bmatrix} 2n+2 \\ n+1 \end{bmatrix}_{q^2} \begin{bmatrix} m+2n+1 \\ m-1 \end{bmatrix}_{-p,q} \begin{bmatrix} 2m+2n+2 \\ m \end{bmatrix}_{p,q} \begin{bmatrix} m+n+1 \\ n+1 \end{bmatrix}_{q^2}^{-1}.
\end{aligned}$$

Example 8 ($n \to 2n+1$ in Corollary 4: $p = \pm q$)

$$\begin{aligned}
&\sum_k (-1)^k \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n+1+2m \\ k+m \end{bmatrix}_{p,q} q^{k^2-2nk-3k} \\
&= \frac{(q;q)_{2n-1}}{(p;q)_{2n-1}} (-1)^{n+1}\, q^{-\left((n+1)^2+1\right)} (1+q)\left(1+pq^{m-1}\right)\left(1-q^{2n+1}\right) \\
&\times \begin{bmatrix} 2n+m-1 \\ n-1 \end{bmatrix}_{q^2} \begin{bmatrix} 2m+2n+1 \\ m \end{bmatrix}_{p,q} \begin{bmatrix} m+2n-1 \\ m \end{bmatrix}_{p,q}^{-1}.
\end{aligned}$$

To illustrate how these formulae can be derived from the Corollaries given in the previous section, we prove now the formula displayed in Example 1 as a showcase.

Choosing $p = q$ and writing $n \to 2n$ in Corollary 2 give us

$$\begin{aligned}&\sum_{k=0}^{2n}(-1)^k \begin{bmatrix}2n\\k\end{bmatrix}_q \begin{bmatrix}2m+2n\\m+k\end{bmatrix}_q q^{k^2-2nk}\\ &= (-q)^{-n^2}\frac{(q;q^2)_n}{(q^{2+2m};q^2)_n}\begin{bmatrix}2m+2n\\m\end{bmatrix}_q \frac{(-q;q)_{m+2n}}{(-q;q)_m}\end{aligned}$$

which can be reduced, by the definition of the q-Pochhammer symbols, to

$$\begin{aligned}&(-1)^n q^{-n^2}\begin{bmatrix}2m+2n\\m\end{bmatrix}_q \frac{(q^2;q^2)_{m+2n}(q;q)_m(q;q)_{2n}}{(q^2;q^2)_{m+n}(q^2;q^2)_n(q;q)_{m+2n}}\\ =&(-1)^n q^{-n^2}\begin{bmatrix}2m+2n\\m\end{bmatrix}_q \begin{bmatrix}m+2n\\m\end{bmatrix}_q^{-1}\begin{bmatrix}m+2n\\n\end{bmatrix}_{q^2}\\ =&(-1)^n q^{-n^2}\begin{bmatrix}2m+2n\\m\end{bmatrix}_q \begin{bmatrix}m+2n\\m\end{bmatrix}_q^{-1}\begin{bmatrix}m+2n\\n\end{bmatrix}_{q^2}.\end{aligned}$$

Similarly for $p = -q$ and $n \to 2n$ in Corollary 2, we have

$$\begin{aligned}&\sum_{k=0}^{2n}(-1)^k \begin{bmatrix}2n\\k\end{bmatrix}_q \begin{bmatrix}2m+2n\\m+k\end{bmatrix}_{-q,q} q^{k^2-2nk}\\ &= (-q)^{-n^2}\frac{(q;q^2)_n}{(q^{2m+2};q^2)_n}\begin{bmatrix}2m+2n\\m\end{bmatrix}_{-q,q} \frac{(q;q)_{m+2n}}{(q;q)_m}.\end{aligned}$$

The right member in the last equality can further be simplified into

$$\begin{aligned}&(-1)^n q^{-n^2}\begin{bmatrix}2m+2n\\m\end{bmatrix}_{-q,q} \frac{(q;q)_{2n}(q;q)_{m+2n}(-q;q)_m}{(q^2;q^2)_{m+n}(q^2;q^2)_n}\\ &(-1)^n q^{-n^2}\frac{(q;q)_{2n}}{(-q;q)_{2n}}\begin{bmatrix}2m+2n\\m\end{bmatrix}_{-q,q} \frac{(-q;q)_m(-q;q)_{2n}(q^2;q^2)_{m+2n}}{(-q;q)_{m+2n}(q^2;q^2)_{m+n}(q^2;q^2)_n}\\ =&(-1)^n q^{-n^2}\frac{(q;q)_{2n}}{(-q;q)_{2n}}\begin{bmatrix}2m+2n\\m\end{bmatrix}_{-q,q} \begin{bmatrix}m+2n\\m\end{bmatrix}_{-q,p}^{-1}\begin{bmatrix}m+2n\\n\end{bmatrix}_{q^2}\end{aligned}$$

which corresponds to the case $p = -q$ of Example 1. The other examples can be derived similarly.

4 Applications to Fibonomial Sums Identities

As described in the introduction, we present, in this section, some applications to the generalized Fibonomial sums identities by specializing the value of $q = \beta/\alpha$, in the examples established in the last section. We point out that all identities displayed below hold for all the nonnegative integers m and n with $\Delta = p^2 + 4$.

$$(a)\quad \sum_k \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U \begin{Bmatrix} 2n+2m \\ k+m \end{Bmatrix}_U U_k U_{rn-k}$$

$$= \frac{U_{2n}U_{rn+m}}{V_{2n+m}} \begin{Bmatrix} 2n+m \\ m \end{Bmatrix}_U^{-1} \begin{Bmatrix} 2n+2m \\ m \end{Bmatrix}_U \begin{Bmatrix} 2n+m \\ n \end{Bmatrix}_{U:2}.$$

$$(b)\quad \sum_k \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U \begin{Bmatrix} 2n+2m \\ k+m \end{Bmatrix}_U = \begin{Bmatrix} 2n+m \\ m \end{Bmatrix}_U^{-1} \begin{Bmatrix} 2n+2m \\ m \end{Bmatrix}_U \begin{Bmatrix} 2n+m \\ n \end{Bmatrix}_{U:2}.$$

$$(c)\quad \sum_k \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U \begin{Bmatrix} 2n+2m \\ k+m \end{Bmatrix}_U V_k U_{n-k}$$

$$= \Delta \frac{U_n^2 U_{n+m}}{V_{2n+m}} \begin{Bmatrix} 2n+m \\ m \end{Bmatrix}_U^{-1} \begin{Bmatrix} 2n+2m \\ m \end{Bmatrix}_U \begin{Bmatrix} 2n+m \\ n \end{Bmatrix}_{U:2}.$$

$$(d)\quad \sum_k \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U \begin{Bmatrix} 2n+2m \\ k+m \end{Bmatrix}_U V_k V_{n-k}$$

$$= \frac{V_n^2 V_{n+m}}{V_{2n+m}} \begin{Bmatrix} 2n+m \\ m \end{Bmatrix}_U^{-1} \begin{Bmatrix} 2n+2m \\ m \end{Bmatrix}_U \begin{Bmatrix} 2n+m \\ n \end{Bmatrix}_{U:2}.$$

$$(e)\quad \sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U \begin{Bmatrix} 2n+2m+1 \\ k+m \end{Bmatrix}_U U_k U_{n+k} = \frac{U_{2n+m+2}U_{3n+1}}{V_m V_{m-1}}$$

$$\times \begin{Bmatrix} 2n+m \\ 2n+2 \end{Bmatrix}_V \begin{Bmatrix} 2n+2 \\ n+1 \end{Bmatrix}_{U:2} \begin{Bmatrix} 2n+2m+2 \\ m \end{Bmatrix}_U \begin{Bmatrix} n+m+1 \\ m \end{Bmatrix}_{U:2}^{-1}.$$

$$(f)\quad \sum_k \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U \left\langle \begin{matrix} 2n+2m \\ k+m \end{matrix} \right\rangle_V U_k U_{rn-k}$$

$$= \Delta^{n-1} \frac{U_{2n}V_{rn+m}}{U_{2n+m}} \left\langle \begin{matrix} 2n+2m \\ m \end{matrix} \right\rangle_V \left\langle \begin{matrix} 2n+m \\ m \end{matrix} \right\rangle_V^{-1} \begin{Bmatrix} 2n+m \\ n \end{Bmatrix}_{U:2} \prod_{i=1}^{2n} \frac{U_i}{V_i}.$$

$$(g)\quad \sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U \left\langle \begin{matrix} 2m+2n \\ m+k \end{matrix} \right\rangle_V$$

$$= \Delta^n \left\langle \begin{matrix} 2m+2n \\ m \end{matrix} \right\rangle_V \left\langle \begin{matrix} m+2n \\ m \end{matrix} \right\rangle_V^{-1} \begin{Bmatrix} m+2n \\ n \end{Bmatrix}_{U:2} \prod_{i=1}^{2n} \frac{U_i}{V_i}.$$

$$(h)\quad \sum_k \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U \left\langle \begin{matrix} 2n+2m \\ k+m \end{matrix} \right\rangle_V V_k U_{n-k}$$

$$= \Delta^n \frac{U_n^2 V_{n+m}}{U_{2n+m}} \begin{Bmatrix} 2n+m \\ m \end{Bmatrix}_V^{-1} \begin{Bmatrix} 2n+2m \\ m \end{Bmatrix}_V \begin{Bmatrix} 2n+m \\ n \end{Bmatrix}_{U:2} \prod_{i=1}^{2n} \frac{U_i}{V_i}.$$

$$(i)\quad \sum_k \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U \left\langle \begin{matrix} 2n+2m \\ k+m \end{matrix} \right\rangle_V V_k V_{n-k}$$

$$= \Delta^n \frac{V_n^2 U_{n+m}}{U_{2n+m}} \left\langle {2m+2n \atop m} \right\rangle_V \left\langle {m+2n \atop m} \right\rangle_V^{-1} \left\{ {m+2n \atop n} \right\}_{U:2} \prod_{i=1}^{2n} \frac{U_i}{V_i}.$$

All the above displayed identities can be proved by converting them into q-series and then making the evaluation, in closed forms, by appealing to the examples given in the last section. We take the identity (a) to exemplify the approach.

First, write explicitly the q-binomial sum corresponding to (a):

$$\begin{aligned}
&\sum_k (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n+2m \\ k+m \end{bmatrix}_q q^{k^2-2kn} \big(1-q^k\big)\big(1-q^{n-k}\big) \\
&= (-1)^n q^{-n^2} \frac{\left(1-q^{2n}\right)\left(1-q^{n+m}\right)}{\left(1+q^{2n+m}\right)} \\
&\times \begin{bmatrix} 2n+m \\ m \end{bmatrix}_q^{-1} \begin{bmatrix} n+m \\ m \end{bmatrix}_q \begin{bmatrix} 2n+2m \\ n+m \end{bmatrix}_q \begin{bmatrix} 2n+m \\ n \end{bmatrix}_{-q} \\
&= (-1)^n q^{-n^2} \frac{\left(1-q^{2n}\right)\left(1-q^{n+m}\right)}{\left(1+q^{2n+m}\right)} \frac{(q)_{2m+2n}\,(q)_{2n}\,(-q)_{2n+m}}{(q)_{2n+m}\,(q^2)_n\,(q^2)_{n+m}},
\end{aligned}$$

where we have reformulated the right member in terms of the q-Pochhammer symbols. Observe that the q-binomial sum on the left can be expressed as

$$\begin{aligned}
&\sum_k (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n+2m \\ k+m \end{bmatrix}_q q^{k^2-2kn} \Big\{ \big(1+q^n\big) - q^k - q^{-k} q^n \Big\} \\
&= (1+q^n) S_1 - S_2 - q^n S_3,
\end{aligned}$$

where

$$\begin{aligned}
S_1 &= \sum_k (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n+2m \\ k+m \end{bmatrix}_q q^{k^2-2kn}, \\
S_2 &= \sum_k (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n+2m \\ k+m \end{bmatrix}_q q^{k^2-2kn+k}, \\
S_3 &= \sum_k (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n+2m \\ k+m \end{bmatrix}_q q^{k^2-2kn-k}.
\end{aligned}$$

Evaluating these three q-binomial sums by Examples 1, 3 and 6, we have the following expression

$$\begin{aligned}
&(1+q^n)\, S_1 - S_2 - q^n S_3 \\
&= (-1)^n (1+q^n)\, q^{-n^2} \frac{(q)_{2m+2n}\,(q)_{2n}\,(-q)_{2n+m}}{(q)_{2n+m}\,(q^2)_n\,(q^2)_{n+m}} \\
&- (-1)^n q^{2n-n^2} (1+q^m) \frac{\left(q^2\right)_{2n}(q)_{2m+2n}\,(-q)_{m+2n-1}}{(q^2)_n\,(q)_{m+2n}\,(q^2)_{n+m}\,(-q)_{2n}}
\end{aligned}$$

$$- (-1)^n q^{n-n^2} \frac{(1+q^m)}{(1+q^{m+2n})} \frac{(q)_{2n} (q)_{2m+2n} (-q)_{m+2n}}{(q)_{m+2n} (q^2)_n (q^2)_{m+n}}$$

which can further be simplified into

$$\begin{aligned}
&(-1)^n q^{-n^2} \frac{(q)_{2m+2n} (q)_{2n} (-q)_{2n+m}}{(q)_{2n+m} (q^2)_n (q^2)_{n+m}} \\
&\times \left\{ 1 + q^n - q^n \frac{(1+q^m)}{(1+q^{m+2n})} - q^{2n} \frac{(1+q^m)}{(1+q^{m+2n})} \right\} \\
&= (-1)^n q^{-n^2} \frac{\left(1-q^{2n}\right)\left(1-q^{m+n}\right)}{(1+q^{m+2n})} \frac{(q)_{2m+2n} (q)_{2n} (-q)_{2n+m}}{(q)_{2n+m} (q^2)_n (q^2)_{n+m}}.
\end{aligned}$$

This completes the proof of the identity (a). □

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