EVALUATION OF SUMS INVOLVING GAUSSIAN q-BINOMIAL COEFFICIENTS WITH RATIONAL WEIGHT FUNCTIONS

EMRAH KILIÇ AND HELMUT PRODINGER

Abstract. We consider sums of the Gaussian q-binomial coefficients with a parametric rational weight function. We use the partial fraction decomposition technique to prove the claimed results. We also give some interesting applications of our results to certain generalized Fibonomial sums weighted with finite products of reciprocal Fibonacci or Lucas numbers.

1. Introduction

Define the second order linear sequences $\{U_n\}$ and $\{V_n\}$ for $n \geq 2$ by

$$U_n = pU_{n-1} + U_{n-2}, \quad U_0 = 0,\, U_1 = 1,$$
$$V_n = pV_{n-1} + V_{n-2}, \quad V_0 = 2,\, V_1 = p.$$

For $n \geq k \geq 1$ and an integer m, define the generalized Fibonomial coefficient with indices in an arithmetic progression by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U;m} := \frac{U_m U_{2m} \dots U_{nm}}{(U_m U_{2m} \dots U_{km})(U_m U_{2m} \dots U_{(n-k)m})}$$

with $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\}_{U;m} = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\}_{U;m} = 1$. When $p = m = 1$, we obtain the usual Fibonomial coefficients, denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_F$. When $m = 1$, we obtain the generalized Fibonomial coefficients, denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_U$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

In this paper, we firstly consider a sum of the Gaussian q-binomial coefficients with a parametric rational weight function of the form: for any positive integer w, any nonzero real number a, nonnegative integer n, integers t and r such that $r \geq -1$

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q (-1)^j q^{\binom{j+1}{2}} \frac{q^{jt}}{(aq^j; q^w)_{r+1}}.$$

We will compute this sum by considering an appropriate partial fraction decomposition.

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Recently, the authors of [2, 3] computed certain Fibonomial sums with generalized Fibonacci and Lucas numbers as coefficients. For example, if n and m are both nonnegative integers, then

$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\} U_{(2m-1)k} = P_{n,m} \sum_{k=1}^{m} \left\{ {2m-1 \atop 2k-1} \right\} U_{(4k-2)n},$$

$$\sum_{k=0}^{2n+1} \left\{ {2n+1 \atop k} \right\} U_{2mk} = P_{n,m} \sum_{k=0}^{m} \left\{ {2m \atop 2k} \right\} U_{(2n+1)2k},$$

$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\} V_{(2m-1)k} = P_{n,m} \sum_{k=1}^{m} \left\{ {2m-1 \atop 2k-1} \right\} V_{(4k-2)n},$$

$$\sum_{k=0}^{2n+1} \left\{ {2n+1 \atop k} \right\} V_{2mk} = P_{n,m} \sum_{k=0}^{m} \left\{ {2m \atop 2k} \right\} V_{(2n+1)2k},$$

where

$$P_{n,m} = \begin{cases} \prod\limits_{k=0}^{n-m} V_{2k} & \text{if } n \geq m, \\ \prod\limits_{k=1}^{m-n-1} V_{2k}^{-1} & \text{if } n < m; \end{cases}$$

alternating analogues of these sums were also evaluated.

We will present here some interesting applications of our results to sums of Fibonomial coefficients with rational weight functions. These kinds of Fibonomial sums involving rational weight functions have not been considered before in the literature, to the best of our knowledge.

Our approach regarding these applications is as follows. We use the Binet forms

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n (1+q^n)$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$ where $\alpha, \beta = (p \pm \sqrt{\Delta})/2$ and $\Delta = p^2 + 4$.

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\left\{ {n \atop k} \right\}_{U;m} = \alpha^{mk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^m} \quad \text{with} \quad q = -\alpha^{-2}.$$

Thus, the evaluations can be done in the "q-world" and then be retranslated into the language of Fibonomial coefficients and the like.

2. The main results

We give our main result on computing Gaussian q-binomial sums with a parametric rational weight function:

Theorem 1. *For any positive integer w, any nonzero real number a, nonnegative integer n, integers t and r such that $r \geq -1$,*

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j+1}{2}+jt}}{(aq^j; q^w)_{r+1}}$$

$$
\begin{aligned}
&= a^{-t}(q;q)_n\Bigg(\sum_{j=0}^{r}\frac{(-1)^j}{(q^w;q^w)_j\,(q^w;q^w)_{r-j}}\frac{q^{w\binom{j+1}{2}-twj}}{(aq^{wj};q)_{n+1}}\\
&\quad+(-1)^{r+1}\sum_{j=0}^{t-r-1}\begin{bmatrix}n+j\\ n\end{bmatrix}_q\begin{bmatrix}t-1-j\\ r\end{bmatrix}_{q^w}q^{w\binom{r+1}{2}+(j-t)rw}a^j\Bigg).
\end{aligned}
$$

Proof. We consider the partial fraction decomposition of the function

$$
h(z)=\frac{1}{(1-z)(1-zq)\dots(1-zq^n)z^{t+1}}\frac{z^{r+1}}{(z-a)(z-aq^w)\dots(z-aq^{rw})}.
$$

The partial fraction decomposition of $h(z)$ takes the form:

$$
\begin{aligned}
h(z)&=\sum_{j=0}^{n}\frac{(-1)^j q^{\binom{j+1}{2}+j(t+1)}}{(aq^j;q^w)_{r+1}\,(1-zq^j)\,(q;q)_j(q;q)_{n-j}}\\
&\quad+\frac{F_0(n,t,a,r)}{z-a}+\frac{F_1(n,t,a,r)}{z-aq^w}+\dots+\frac{F_r(n,t,a,r)}{z-aq^{rw}}\\
&\quad+\frac{G_1(n,t,a,r)}{z}+\dots+\frac{G_{t-r}(n,t,a,r)}{z^{t-r}}.
\end{aligned}
$$

Now we multiply this relation by z and let $z\to\infty$ and obtain

$$
\begin{aligned}
0=\lim_{z\to\infty}\Bigg(&\sum_{j=0}^{n}\frac{(-1)^j q^{\binom{j+1}{2}+j(t+1)}}{(aq^j;q^w)_{r+1}\,(q;q)_j(q;q)_{n-j}}\frac{z}{(1-zq^j)}\\
&+F_0(n,t,a,r)\frac{z}{z-a}+F_1(n,t,a,r)\frac{z}{z-aq^w}\\
&+\dots+F_r(n,t,a,r)\frac{z}{z-aq^{rw}}+G_1(n,t,a,r)\Bigg),
\end{aligned}
$$

which gives us the equation

$$
0=\sum_{j=0}^{n}\frac{(-1)^j q^{\binom{j+1}{2}+j(t+1)}}{(aq^j;q^w)_{r+1}\,(-q^j)\,(q;q)_j(q;q)_{n-j}}+\sum_{j=0}^{r}F_j\,(n,t,a,r)+G_1(n,t,a,r)
$$

or

$$
\sum_{j=0}^{n}\frac{(-1)^j q^{\binom{j+1}{2}+jt}}{(aq^j;q^w)_{r+1}\,(q;q)_j(q;q)_{n-j}}=\sum_{j=0}^{r}F_j(n,t,a,r)+G_1(n,t,a,r)
$$

where

$$
\begin{aligned}
F_k(n,t,a,r)=\Bigg(&\frac{1}{(1-z)(1-zq)\dots(1-zq^n)z^{t+1}}\\
&\times\frac{z^{r+1}}{(z-a)\dots(z-aq^{(k-1)w})(z-aq^{(k+1)w})\dots(z-aq^{rw})}\Bigg)\Bigg|_{z=aq^{kw}}
\end{aligned}
$$

and

$$
G_1(n,t,a,r)=[z^{-1}]h(z).
$$

First we work out $F_k(n,t,a,r)$:

$$\begin{aligned}
F_k(n,t,a,r) &= \frac{a^{r-t}q^{kw(r-t)}}{(1-aq^{kw})(1-aq^{kw+1})\dots(1-aq^{kw+n})}\\
&\times \frac{1}{(aq^{kw}-a)(aq^{kw}-aq^{w})\dots(aq^{kw}-aq^{(k-1)w})}\\
&\times \frac{1}{(aq^{kw}-aq^{(k+1)w})(aq^{kw}-aq^{(k+2)w})\dots(aq^{kw}-aq^{rw})}\\
&= \frac{a^{r-t}q^{kw(r-t)}}{(aq^{kw};q)_{n+1}}\\
&\times \frac{(-1)^k}{a^k q^{\frac{k(k-1)}{2}w}(1-q^{kw})(1-q^{(k-1)w})\dots(1-q^{w})}\\
&\times \frac{1}{(aq^{kw})^{r-k}(1-q^{w})(1-q^{2w})\dots(1-q^{(r-k)w})}\\
&= \frac{a^{-t}q^{kw(r-t)-\frac{k(k-1)}{2}w-kw(r-k)}(-1)^k}{(aq^{kw};q)_{n+1}}\frac{1}{(q^w;q^w)_k}\frac{1}{(q^w;q^w)_{r-k}}\\
&= \frac{a^{-t}q^{-kwt+\frac{k(k+1)}{2}w}(-1)^k}{(aq^{kw};q)_{n+1}}\frac{1}{(q^w;q^w)_k}\frac{1}{(q^w;q^w)_{r-k}}.
\end{aligned}$$

Now we compute $G_1(n,t,a,r)$:

$$\begin{aligned}
G_1(n,t,a,r) &= [z^{-1}]h(z)\\
&= [z^{-1-r+t}]\frac{1}{(1-z)(1-zq)\dots(1-zq^n)}\frac{1}{(z-a)(z-aq^w)\dots(z-aq^{rw})}\\
&= [z^{-1-r+t}]\sum_{k\ge 0}\begin{bmatrix}n+k\\k\end{bmatrix}_q z^k\cdot\frac{1}{a^{r+1}q^{\frac{r(r+1)}{2}w}(-1)^{r+1}}\sum_{l\ge 0}\begin{bmatrix}r+l\\l\end{bmatrix}_{q^w}\left(\frac{z}{aq^{rw}}\right)^l\\
&= \sum_{l=0}^{t-1-r}\begin{bmatrix}r+t-1-r-l\\r\end{bmatrix}_{q^w}a^{-(t-r-1-l)}q^{-rw(t-1-r-l)}\begin{bmatrix}n+l\\n\end{bmatrix}_q\frac{(-1)^{r+1}}{a^{r+1}q^{\frac{r(r+1)}{2}w}}\\
&= \sum_{l=0}^{t-1-r}\begin{bmatrix}r+t-1-r-l\\r\end{bmatrix}_{q^w}\begin{bmatrix}n+l\\n\end{bmatrix}_q a^{l-t}q^{-rw(t-1-r-l)-\frac{r(r+1)}{2}w}(-1)^{r+1}\\
&= (-1)^{r-1}\sum_{l=0}^{t-r-1}\begin{bmatrix}n+l\\n\end{bmatrix}_q\begin{bmatrix}t-1-l\\r\end{bmatrix}_{q^w}q^{w\binom{r+1}{2}+(l-t)rw}a^{l-t}.
\end{aligned}$$

Therefore we write

$$\begin{aligned}
&\sum_{j=0}^{n}\frac{(-1)^j q^{\binom{j+1}{2}+jt}}{(aq^j;q^w)_{r+1}(q;q)_j(q;q)_{n-j}}\\
&= a^{-t}\sum_{j=0}^{r}\frac{(-1)^j}{(q^w;q^w)_j(q^w;q^w)_{r-j}}\frac{q^{w\binom{j+1}{2}-twj}}{(aq^{wj};q)_{n+1}}
\end{aligned}$$

$$+ a^{-t}(-1)^{r+1} \sum_{j=0}^{t-r-1} \begin{bmatrix} n+j \\ n \end{bmatrix}_q \begin{bmatrix} t-1-j \\ r \end{bmatrix}_{q^w} q^{w\binom{r+1}{2}+(j-t)rw} a^j.$$

Multiplying both sides of the above equation with $(q;q)_n$, we have the claimed result. □

As a consequence of Theorem 1, we have the following Corollary.

Corollary 1. *For any positive integer w, any nonzero real number a, nonnegative integer n, integers t and r such that $r \geq -1$ and $t < r+1$,*

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-1)^j q^{\binom{j+1}{2}+jt}}{(aq^j;q^w)_{r+1}} = a^{-t}(q;q)_n \sum_{j=0}^{r} \frac{(-1)^j}{(q^w;q^w)_j (q^w;q^w)_{r-j}} \frac{q^{w\binom{j+1}{2}-twj}}{(aq^{wj};q)_{n+1}}.$$

3. Applications

Now we will give some interesting corollaries of our main result to Fibonomial sums. We start with a Fibonomial-Lucas corollary :

Corollary 2.

$$\sum_{k=0}^{4n+1} \begin{Bmatrix} 4n+1 \\ k \end{Bmatrix} \frac{1}{V_k V_{k+1}} (-1)^{k(k-1)/2} = \frac{1}{2} \frac{U_{4n+1}!}{V_{4n+1}!} \Delta^{2n+1} U_{2n+1}^2,$$

where Δ is defined as before and the generalized Fibonacci and Lucas factorials are $U_n! = U_n U_{n-1} \ldots U_1$ and $V_n! = V_n V_{n-1} \ldots V_1$, respectively.

Proof. To prove the identity, if we convert it into q-notation, then we must prove that

$$\sum_{j=0}^{4n+1} \begin{bmatrix} 4n+1 \\ j \end{bmatrix}_q \frac{(-1)^j q^{\frac{1}{2}j(j+1)} q^{-2jn}}{(1+q^{j+1})(1+q^j)} = \frac{1}{2} \frac{(q;q)_{4n+1}}{(-q;q)_{4n+2}} \frac{(1-q^{2n+1})^2}{(1-q)}.$$

If we take $n \to 4n+1$, $a = -q$ and $r = w = 1$ in Theorem 1, then we write

$$\sum_{j=0}^{4n} \begin{bmatrix} 4n+1 \\ j \end{bmatrix}_q (-1)^j q^{\binom{j+1}{2}} \frac{q^{jt}}{(-q^j;q)_{r+1}}$$

$$= (-1)^t (q;q)_{4n+1} \sum_{j=0}^{1} \frac{(-1)^j}{(q;q)_j (q;q)_{1-j}} \frac{q^{\binom{j+1}{2}-tj}}{(-q^j;q)_{4n+2}}$$

$$+ (-1)^t (q;q)_n \sum_{j=0}^{t-2} \begin{bmatrix} 4n+1+j \\ n \end{bmatrix}_q \begin{bmatrix} t-1-j \\ r \end{bmatrix}_q (-1)^j q^{1+j-t}.$$

If we choose $t = -2n$ where n is nonnegative integer, then the condition $t < r+1$ is satisfied for $r = 1$ and so we write the last equation as

$$\sum_{j=0}^{4n} \begin{bmatrix} 4n+1 \\ j \end{bmatrix}_q (-1)^j q^{\binom{j+1}{2}} \frac{q^{-2nj}}{(-q^j;q)_{r+1}}$$

$$= (q;q)_{4n+1} \sum_{j=0}^{1} \frac{(-1)^j}{(q;q)_j (q;q)_{1-j}} \frac{q^{\binom{j+1}{2}+2nj}}{(-q^j;q)_{4n+2}}$$

which equals

$$= (q;q)_{4n+1}\Big(\frac{1}{(q;q)_1(-1;q)_{4n+2}} - \frac{q^{2n+1}}{(q;q)_1(-q;q)_{4n+2}}\Big)$$

$$= (q;q)_{4n+1}\frac{1}{(q;q)_1(-1;q)_{4n+3}}\Big(1+q^{4n-2}-2q^{2n+1}\Big)$$

$$= \frac{1}{2}\frac{(q;q)_{4n+1}}{(q;q)_1}\frac{(1-q^{2n+1})^2}{(-q;q)_{4n+2}},$$

as claimed. □

Corollary 3. *For $n>0$,*

$$\sum_{j=0}^{n} \begin{Bmatrix} n \\ j \end{Bmatrix}_U \frac{1}{U_{j+1}U_{j+2}} (-1)^{\frac{1}{2}j(j+1)} \alpha^{-j(n+1)} = \frac{\alpha^n}{F_{n+2}},$$

where α is defined as before.

Proof. First we convert the claim into q-notation. Consider

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{\alpha^{j(n-j)}(-1)^{\frac{1}{2}j(j+1)}\alpha^{-j(n+1)}}{\alpha^j\alpha^{j+1}(1-q^{j+1})(1-q^{j+2})} = \frac{\alpha^n}{\alpha^{n+1}(1-q^{n+2})(1-q)}$$

or

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \alpha^{-j^2-3j-1} \frac{(-1)^{\frac{1}{2}j(j+1)}}{(1-q^{j+1})(1-q^{j+2})} = \frac{\alpha^n}{\alpha^{n+1}(1-q^{n+2})(1-q)}$$

or

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{\alpha^{-j^2-3j}(-1)^{\frac{1}{2}j(j+1)}}{(1-q^{j+1})(1-q^{j+2})} = \frac{1}{(1-q^{n+2})(1-q)}$$

or

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(-1)^j q^{\frac{1}{2}j(j+3)}}{(1-q^{j+1})(1-q^{j+2})} = \frac{1}{(1-q^{n+2})(1-q)},$$

which follows from Corollary 1 by taking $t=r=w=1$ and $a=q$. □

By using Corollary 1, we obtain the following result:

Corollary 4. *Let n and l be any nonnegative integers and $c \in \{0,1,2,3\}$.*

$$\sum_{k=0}^{4n-c} \begin{Bmatrix} 4n-c \\ k \end{Bmatrix}_U \frac{U_k^{4l+r+1}}{V_{k-1}V_{k+1}} (-1)^{k(k-1)/2} = -\frac{1}{2}\frac{U_{4n-c+1}!}{V_{4n-c+1}!}\Delta^{2n-2l-r}U_{4n-c}U_2^{4l+c-1},$$

where Δ is defined as before, and $n \geq l$ for $c \in \{0,3\}$, and $n > l$ for $c \in \{1,2\}$.

Proof. We start by converting the claim into q-notation. Thus we must prove that

$$\sum_{k=0}^{4n-c} \begin{bmatrix} 4n-c \\ k \end{bmatrix}_q \frac{\left(1-q^k\right)^{4l+c+1}}{(1+q^{k+1})(1+q^{k-1})} (-1)^k q^{\frac{1}{2}k(k-4l-4n+1)}$$

$$= \frac{1}{2} q^{1-2(n+l)} \frac{\left(1-q^{4n-c}\right)(1+q)^{4l+c-1}}{(1-q)} \frac{(q;q)_{4n-c+1}}{(-q;q)_{4n-c+1}},$$

which by the binomial theorem equals

$$\sum_{k=0}^{4n-c} \begin{bmatrix} 4n-c \\ k \end{bmatrix}_q \frac{(-1)^k q^{\frac{1}{2}k(k+1)} q^{-2k(l+n)}}{\left(1+q^{k+1}\right)\left(1+q^{k-1}\right)} \sum_{s=0}^{4l+c+1} \binom{4l+c+1}{s} (-1)^s q^{ks}$$
$$= \frac{1}{2} q^{1-2(n+l)} \frac{\left(1-q^{4n-c}\right)(1+q)^{4l+c-1}}{(1-q)} \frac{(q;q)_{4n-c+1}}{(-q;q)_{4n-c+1}},$$

which, by changing the summation order, equals

$$\sum_{s=0}^{4l+c+1} \binom{4l+c+1}{s} (-1)^s \sum_{k=0}^{4n-c} \begin{bmatrix} 4n-c \\ k \end{bmatrix}_q (-1)^k \frac{q^{\frac{1}{2}k(k+1)} q^{k(s-2l-2n)}}{\left(1+q^{k+1}\right)\left(1+q^{k-1}\right)}$$
$$= \frac{1}{2} q^{1-2(n+l)} \frac{\left(1-q^{4n-c}\right)(1+q)^{4m+c-1}}{(1-q)} \frac{(q;q)_{4n-c+1}}{(-q;q)_{4n-c+1}}.$$

Now we consider the sums in LHS of the claim just above:

$$\sum_{k=0}^{4n-c} \begin{bmatrix} 4n-c \\ k \end{bmatrix}_q (-1)^k q^{\frac{1}{2}k(k+1)} q^{k(s-2l-2n)} \frac{1}{\left(1+q^{k+1}\right)\left(1+q^{k-1}\right)},$$

where $0 \leq s \leq 4l+c+1$. For $r=1$, $w=2$, $a=-q^{-1}$, $t=s-2l-2n$, the hypothesis of Theorem 1 is satisfied, that is, $t<r+1$, and so we write by Corollary 1,

$$\sum_{j=0}^{4n-c} \begin{bmatrix} 4n-c \\ j \end{bmatrix}_q (-1)^j q^{\binom{j+1}{2}} \frac{q^{j(s-2l-2n)}}{\left(-q^{j-1};q^2\right)_2}$$
$$= (-1)^{(s-2l-2n)} (q;q)_{4n-c} \sum_{j=0}^{1} \frac{(-1)^j}{\left(q^2;q^2\right)_j \left(q^2;q^2\right)_{1-j}} \frac{q^{j(j+1)-(s-2l-2n)(2j-1)}}{\left(-q^{2j-1};q\right)_{4n-c+1}}$$

or

$$\sum_{j=0}^{4n-c} \begin{bmatrix} 4n-c \\ j \end{bmatrix}_q (-1)^j q^{\binom{j+1}{2}} \frac{q^{j(s-2l-2n)}}{\left(1+q^{j-1}\right)\left(1+q^{j+1}\right)}$$
$$= (-1)^s (q;q)_{4n-c} \left(\frac{1}{\left(q^2;q^2\right)_1} \frac{q^{s-2l-2n}}{\left(-q^{-1};q\right)_{4n-c+1}} - \frac{1}{\left(q^2;q^2\right)_1} \frac{q^{2-(s-2l-2n)}}{(-q;q)_{4n-c+1}} \right)$$
$$= (-1)^s (q;q)_{4n-c} \frac{1}{\left(q^2;q^2\right)_1} \left(\frac{q^{s-2l-2n}}{\left(-q^{-1};q\right)_{4n-c+1}} - \frac{q^{2-(s-2l-2n)}}{(-q;q)_{4n-c+1}} \right).$$

From the last equation, we write

$$\sum_{s=0}^{4l+c+1} \binom{4l+c+1}{s} (-1)^s \sum_{j=0}^{4n-c} \begin{bmatrix} 4n-c \\ j \end{bmatrix}_q (-1)^j q^{\binom{j+1}{2}} \frac{q^{j(s-2l-2n)}}{\left(1+q^{j-1}\right)\left(1+q^{j+1}\right)}$$
$$= (q;q)_{4n-c} \frac{1}{\left(q^2;q^2\right)_1}$$

$$\times\left(\frac{q^{-2l-2n}}{(-q^{-1};q)_{4n-c+1}}\sum_{s=0}^{4l+c+1}\binom{4l+c+1}{s}q^{s}-\frac{q^{2-(-2l-2n)}}{(-q;q)_{4n-c+1}}\sum_{s=0}^{4l+c+1}\binom{4l+c+1}{s}q^{-s}\right)$$

$$=(q;q)_{4n-c}\frac{1}{(q^2;q^2)_1}\left(\frac{q^{-2l-2n}(1+q)^{4l+c+1}}{(-q^{-1};q)_{4n-c+1}}-\frac{q^{2-(-2l-2n)}q^{-(4l+c+1)}(1+q)^{4l+c+1}}{(-q;q)_{4n-c+1}}\right)$$

$$=(q;q)_{4n-c}\frac{(1+q)^{4l+c+1}}{(q^2;q^2)_1}\left(\frac{q^{-2l-2n}}{(-q^{-1};q)_{4n-c+1}}-\frac{q^{2n-2l-c+1}}{(-q;q)_{4n-c+1}}\right)$$

$$=(q;q)_{4n-c}\frac{q^{-2l}(1+q)^{4l+c+1}}{(q^2;q^2)_1}\left(\frac{q^{-2n}}{(-q^{-1};q)_{4n-c+1}}-\frac{q^{2n-c+1}}{(-q;q)_{4n-c+1}}\right)$$

$$=(q;q)_{4n-c}\frac{q^{-2l}(1+q)^{4l+c+1}}{(q^2;q^2)_1}\left(\frac{q^{-2n}(1+q^{4n-c})(1+q^{4n-c+1})-2(1+q)q^{2n-c+1}}{2(1+q)(-q;q)_{4n-c+1}}\right)$$

$$=(q;q)_{4n-c}\frac{q^{-2l}(1+q)^{4l+c+1}}{(q^2;q^2)_1}\frac{q^{-2n+1}(1-q^{4n-c+1})(1-q^{4n-c})}{2(1+q)(-q;q)_{4n-c+1}}$$

$$=\frac{1}{2}q^{1-2n-2l}\frac{(1-q^{4n-c})(1+q)^{4l+c-1}}{(1-q)}\frac{(q;q)_{4n-c+1}}{(-q;q)_{4n-c+1}},$$

as claimed. □

Corollary 5. *For $n>0$,*

$$\sum_{j=0}^{4n+1}\begin{Bmatrix}4n+1\\ j\end{Bmatrix}_U\frac{1}{U_{j+1}U_{j+3}}(-1)^{j(j-1)/2}=\frac{1}{V_1U_{4n+2}}-\frac{1}{U_{4n+2}U_{4n+3}U_{4n+4}}.$$

Proof. First we convert the claimed identity into to q-notation. Thus we must prove that

$$\sum_{j=0}^{4n+1}\begin{bmatrix}4n+1\\ j\end{bmatrix}_q(-1)^j q^{\binom{j+1}{2}}\frac{q^{-2nj}}{(1-q^{j+1})(1-q^{j+3})}=\frac{(q;q)_{4n+1}}{(1-q^2)}\left(\frac{q^{2n}}{(q;q)_{4n+2}}-\frac{q^{2+6n}}{(q^3;q)_{4n+2}}\right).$$

In order to confirm this identity, it is enough to take $n\to 4n+1$, $r=1$, $w=2$, $a=q$ and $t=-2n$ in Corollary 1. □

References

[1] G. E. Andrews, R. Askey, R. Roy, *Special functions*, Cambridge University Press (2000).

[2] E. Kılıç, H. Prodinger, I. Akkus, H. Ohtsuka, "Formulas for Fibonomial Sums with generalized Fibonacci and Lucas coefficients", *The Fibonacci Quarterly*, **49**:4 (2011), 320–329.

[3] E. Kılıç, H. Ohtsuka, I. Akkus, "Some generalized Fibonomial sums related with the Gaussian q-binomial sums", *Bull. Math. Soc. Sci. Math. Roumanie*, **55**:103 No. 1 (2012), 51–61.

TOBB University of Economics and Technology Mathematics Department 06560 Ankara Turkey

E-mail address: ekilic@etu.edu.tr

Department of Mathematics, University of Stellenbosch 7602 Stellenbosch South Africa

E-mail address: hproding@sun.ac.za