On sums of squares of Fibonomial coefficients by q-calculus

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Abstract

We present some new kinds of sums of squares of Fibonomial coefficients with finite products of generalized Fibonacci and Lucas numbers as coefficients. As proof method, we will follow the method given in [8]. For this, first we translate everything into q-notation, and then to use generating functions and Rothe's identity from classical q-calculus.

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1 Introduction

Define the second order linear sequences $\{U_n\}$ and $\{V_n\}$ for $n \geq 2$ by

$$U_n = pU_{n-1} + U_{n-2}, \quad U_0 = 0,\ U_1 = 1,$$
$$V_n = pV_{n-1} + V_{n-2}, \quad V_0 = 2, \quad V_1 = p.$$

When $p = 1$, we obtain the usual Fibonacci numbers F_n and Lucas numbers L_n, respectively.

For $n \geq k \geq 1$ and an integer m, define the generalized Fibonomial coefficient with indices in an arithmetic progression by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U;m} := \frac{U_m U_{2m} \dots U_{nm}}{(U_m U_{2m} \dots U_{km})(U_m U_{2m} \dots U_{(n-k)m})}$$

with $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\}_{U;m} = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\}_{U;m} = 1$. When $p = m = 1$, we obtain the usual Fibonomial coefficients, denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_F$. When $m = 1$, we obtain the generalized Fibonomial coefficients, denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_U$. As an interesting generalization of the binomial coefficients, the Fibonomial coefficients have taken the interest of several authors. For their properties, we refer to [2, 3, 4, 5, 11].

A special case is the nth central generalized Fibonomial coefficient with indices in an arithmetic progression, defined as $\left\{ \begin{smallmatrix} 2n \\ n \end{smallmatrix} \right\}_{U;m}$. When $m = p = 1$, we obtain the nth central Fibonomial coefficient, denoted by $\left\{ \begin{smallmatrix} 2n \\ n \end{smallmatrix} \right\}_F$. Our evaluations will be in terms of such numbers.

Our approach is as follows. We use the Binet forms

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{1 - q^n}{1 - q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n (1 + q^n)$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$ where $\alpha, \beta = (p \pm \sqrt{\Delta})/2$ and $\Delta = p^2 + 4$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U;m} = \alpha^{mk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^m} \quad \text{with} \quad q = -\alpha^{-2}.$$

We recall that one version of the *Cauchy binomial theorem* is given by

$$\sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = \prod_{k=1}^{n} (1 + xq^k),$$

and *Rothe's* formula [1] is

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (x;q)_n = \prod_{k=0}^{n-1} (1 - xq^k).$$

The authors of [6, 9, 7] computed certain Fibonomial sums with generalized Fibonacci and Lucas numbers as coefficients. For example, if n and m are both nonnegative integers, then

$$\sum_{k=0}^{2n} \left\{ \begin{matrix} 2n \\ k \end{matrix} \right\} U_{(2m-1)k} = P_{n,m} \sum_{k=1}^{m} \left\{ \begin{matrix} 2m-1 \\ 2k-1 \end{matrix} \right\} U_{(4k-2)n},$$

where

$$P_{n,m} = \begin{cases} \prod_{k=0}^{n-m} V_{2k} & \text{if } n \geq m, \\ \prod_{k=1}^{m-n-1} V_{2k}^{-1} & \text{if } n < m; \end{cases}$$

alternating analogues of these sums were also presented. In particular, if m is a small number, we can think about this as the closed form evaluation of the left-hand side in terms of the finitely many terms of the right-hand side.

The authors [10] gave some sums formulae including Fibonomial coefficients, Fibonacci and Lucas numbers. For example, for positive integers m and n, they showed that

$$\sum_{j=0}^{4m+2} (-1)^{\frac{j(j-1)}{2}} \left\{ \begin{matrix} 4m \\ j \end{matrix} \right\} F_{n+4m-j} = \frac{1}{2} F_{2m+n} \sum_{j=0}^{4m} (-1)^{\frac{j(j-1)}{2}} \left\{ \begin{matrix} 4m \\ j \end{matrix} \right\} L_{2m-j},$$

The authors [8] gave a systematic approach to compute certain sums of squares of Fibonomial coefficients with finite products of generalized Fibonacci and Lucas numbers as coefficients. For example, if n is nonnegative integer, then they proved the following Gaussian q-binomial sums identity

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 (-1)^k q^{k^2-2kn-3k} (1-q^{2k})^2$$
$$= 2(-1)^{n+1} q^{-n^2-2n-2} \frac{(1+q)(1-q^{2n+1})(1-q^{2n+1})}{(1+q^{2n})} \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_{q^2}.$$

In this paper, we present some new classes of sums formulae including squares of the Fibonomial coefficients with finite product of generalized Fibonacci and Lucas numbers as coefficients. We translate everything into q-notation, and then to use generating functions and Rothe's identity from classical q-calculus as described in [8].

All the identities, we will derive, hold for general q, and results about generalized Fibonacci and Lucas numbers come out as corollaries for the special choice of q. We will frequently denote $\left\{ {n \atop k} \right\}_{U;1}$ by $\left\{ {n \atop k} \right\}_U$.

In the next section, we list our generalized Fibonomial-Fibonacci-Lucas sums identities and then, in the Section 3, we present their q-versions, resp. In the last Section, we prove the first identity as showcase to show how our method is applied in practice.

2 The main results

We present our sums formulae including squares of the Fibonomial coefficients. For nonnegative integers n and m,

1. $$\sum_{k=0}^{2n+1} \left\{ {2n+1 \atop k} \right\}_U^2 U_{2k} V_{m(2n+1)+2k} (-1)^k = -2U_2 U_{2n+1} V_{(m+2)(2n+1)} \left\{ {2n-1 \atop n} \right\}_{U;2},$$

2. $$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\}_U^2 U_{2n-k} U_{2n+k} = U_{2n} U_{4n} \left\{ {2n-1 \atop n} \right\}_{U;2},$$

3. $$\sum_{k=0}^{2n+1} \left\{ {2n+1 \atop k} \right\}_U^2 U_{2n-k+1}^2 U_{m(2n+1)+k}^2 (-1)^k = 2U_{2n+1}^2 U_{m(2n+1)} U_{(m+1)(2n+1)-1} \left\{ {2n-1 \atop n} \right\}_{U;2},$$

4. $$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\}_U^2 U_{2n-k}^2 U_{2nm+k}^2 U_k^2 (-1)^k = -2\frac{U_{2nm+1} U_{2n-1} U_{2n}^3 U_{(m+1)(2n)-1}}{V_{2n-1} V_{2n-2}} \left\{ {2n-1 \atop n} \right\}_{U;2},$$

5. $$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\}_U^2 U_{2nm+2k} (-1)^k = 2U_{2(m+1)n} \left\{ {2n-1 \atop n} \right\}_{U;2},$$

6. $$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\}_U^2 U_{2nm-k}^2 (-1)^k = 2U_{2nm} U_{2(m-1)n} \left\{ {2n-1 \atop n} \right\}_{U;2},$$

7. $$\sum_{k=0}^{2n+1} \left\{ {2n+1 \atop k} \right\}_U^2 U_{(2n+1)m+2k}^2 (-1)^k = -2U_2 U_{2n+1} U_{2(m+1)(2n+1)} \left\{ {2n-1 \atop n} \right\}_{U;2},$$

8. $$\sum_{k=0}^{2n+1} \left\{ {2n+1 \atop k} \right\}_U^2 V_{2n+1-k} V_{m(2n+1)+k} (-1)^k = -\Delta U_{2n+1} V_{2n} U_{m(2n+1)} \left\{ {2n-1 \atop n} \right\}_{U;2},$$

9. $$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\}_U^2 U_k V_{2nm+k} (-1)^k = U_{2n} V_{2nm} \left\{ {2n-1 \atop n} \right\}_{U;2},$$

10.

$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U^2 V_{2n+1-2k} V_{m(2n+1)+2k} (-1)^k = 2\Delta U_2 U_{2n+1} U_{(m+1)(2n+1)} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

11.

$$\sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U^2 U_{2(n-k)} V_{2(nm+k)} = -\Delta \frac{U_2 U_{2n}^2 U_{(m+1)2n}}{V_{2n-1}} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

12.

$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U^2 U_{2n+1-k}^2 V_{m(2n+1)+k}^2 (-1)^k = 2U_{2n+1}^2 V_{m(2n+1)} V_{(m+1)(2n+1)-1} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

13.

$$\sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U^2 U_{2n-k}^2 V_{2nm+k}^2 = \Delta \frac{U_{2n}^3 U_{(2m+1)2n-1}}{V_{2n-1}} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

14.

$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U^2 U_{2n+1-k}^2 V_{(2n+1)m+k}^2 U_k^2 = \Delta \frac{U_{2n+1}^2 U_{2n}^3 U_{(2m+1)(2n+1)}}{V_{2n-1}} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

15.

$$\sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U^2 V_{2(nm+k)} (-1)^k = 2V_{2(m+1)n} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

16.

$$\sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U^2 U_{2n+k}^2 (-1)^k = 2U_{4n} U_{4n-2} \begin{Bmatrix} 2n-2 \\ n-1 \end{Bmatrix}_{U;2},$$

17.

$$\sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U^2 U_k U_{2n+k} (-1)^k = U_{2n}^2 \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

18.

$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U^2 U_{2(n+k)+3} U_{2(n+k)+1} (-1)^k = -2U_2 U_{8n+6} U_{2n+1} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

19.

$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U^2 U_{2k} U_{2(n+k)+1} (-1)^k = -2U_2 U_{3(2n+1)} U_{2n+1} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

20.

$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U^2 U_{2n-k+1} U_{2n+k+1} (-1)^k = U_{2n+1}^2 V_{2n} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

21.

$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U^2 U_{2(n-k)+1} U_{2(n+k)+1} (-1)^k = -2U_2 U_{2n+1} U_{4n+2} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

22.

$$\sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U^2 U_{2(n-k)} U_{2(n+k)} = -\frac{U_2 U_{2n}^2 V_{4n}}{V_{2n-1}} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

23.
$$\sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U^2 U_{2n-k}^2 U_{2n+k}^2 = \frac{U_{6n-1} U_{2n}^3}{V_{2n-1}} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

24.
$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U^2 U_{2n-k+1}^2 U_{2n+k+1}^2 U_k^2 = \frac{U_{3(2n+1)} U_{2n+1}^2 U_{2n}^3}{V_{2n-1}} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

25.
$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U^2 U_{2n+1-k}^2 U_{2n+1+k}^2 U_{2k}^2 (-1)^k = 2\frac{U_3 V_{4n+2} U_{2n+1}^3 U_{2n}^2 U_{4n}}{V_{2n-2} V_{2n-1}} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

26.
$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U^2 U_{2n+1-k} U_{2n+1+k} U_k U_{3k} (-1)^k = \frac{U_2^2 U_{6n+3} U_{2n+1}^2 U_{2n}}{V_{2n-1}} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

27.
$$\sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U^2 U_{2n-3k} U_{2n+k} (-1)^k = -\Delta \frac{U_{2n}^3 U_{2n+1}}{V_{2n-1}} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

28.
$$\sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U^2 V_{2n-k} V_{2n+k} = V_{2n}^3 \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

29.
$$\sum_{k=0}^{2n} \begin{Bmatrix} 2n \\ k \end{Bmatrix}_U^2 U_{2n-k}^2 V_{2n+k}^2 U_k^2 (-1)^k = -2\frac{U_{2n}^3 U_{2n-1} V_{2n+1} V_{4n-1}}{V_{2n-1} V_{2n-2}} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

30.
$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U^2 U_{2n+1-k} V_{2n+1+k} U_k V_{3k} (-1)^k = \Delta \frac{U_2^2 U_{2n+1}^2 U_{2n} U_{6n+3}}{V_{2n-1}} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2},$$

where Δ is defined as before.

3 The q-version of the identities

Now we give q-version of the identities given in the previous section. For nonnegative integers n and m,

1.
$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 (-1)^k q^{k^2-2kn-3k} \left(1-q^{2k}\right) \left(1+q^{(2n+1)m+2k}\right)$$
$$= 2(-1)^{n+1} q^{-\left(n^2+2n+2\right)} \left(1-q^{2n+1}\right)(1+q)\left(1+q^{(m+2)(2n+1)}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

2.
$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 q^{k^2-2kn} \left(1-q^{2n-k}\right)\left(1-q^{2n+k}\right)(-1)^k = (-1)^n q^{-n^2} \left(1-q^{2n}\right)\left(1-q^{4n}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

3.

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \left(1-q^{2n-k+1}\right)^2 \left(1-q^{m(2n+1)+k}\right)^2 q^{k^2-2kn-k} \left(-1\right)^k$$

$$= 2\left(-1\right)^n q^{-n^2} \left(1-q^{2n+1}\right)^2 \left(1-q^{m(2n+1)}\right) \left(1-q^{(m+1)(2n+1)-1}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

4.

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \left(1-q^{2n-k}\right)^2 \left(1-q^{2nm+k}\right)^2 \left(1-q^k\right)^2 q^{k^2-2kn-k} \left(-1\right)^k$$

$$= 2\left(-1\right)^n \frac{q^{-n^2-1}\left(1-q^{2nm+1}\right)\left(1-q^{2n-1}\right)\left(1-q^{2n}\right)^3\left(1-q^{(m+1)(2n)-1}\right)}{\left(1+q^{2n-1}\right)\left(1+q^{2n-2}\right)} \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

5.

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \left(1-q^{2nm+2k}\right) q^{k^2-2kn-k} \left(-1\right)^k = 2\left(-1\right)^n q^{-n^2} \left(1-q^{2(m+1)n}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

6.

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \left(1-q^{2nm-k}\right)^2 q^{k-2kn+k^2} \left(-1\right)^k = 2\left(-1\right)^n q^{-n(n-2)} \left(1-q^{2nm}\right) \left(1-q^{2(m-1)n}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

7.

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \left(1-q^{m(2n+1)+2k}\right)^2 q^{k^2-2kn-3k} \left(-1\right)^k$$

$$= \left(-1\right)^{n+1} 2q^{-n^2-2n-2} \left(1+q\right) \left(1-q^{2n+1}\right) \left(1-q^{2(m+1)(2n+1)}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

8.

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \left(1+q^{2n-k+1}\right) \left(1+q^{m(2n+1)+k}\right) q^{k^2-2kn-k} \left(-1\right)^k$$

$$= \left(-1\right)^{n+1} q^{-n^2} \left(1-q^{2n+1}\right) \left(1+q^{2n}\right) \left(1-q^{m(2n+1)}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

9.

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \left(1-q^k\right) \left(1+q^{2nm+k}\right) q^{k^2-2kn-k} \left(-1\right)^k = \left(-1\right)^n q^{-n^2} \left(1+q^{2nm}\right) \left(1-q^{2n}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

10.

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \left(1+q^{2n+1-2k}\right) \left(1+q^{m(2n+1)+2k}\right) q^{k^2-2kn-k} \left(-1\right)^k$$

$$= 2\left(-1\right)^{n+1} q^{-n^2-1} \left(1+q\right) \left(1-q^{2n+1}\right) \left(1-q^{(m+1)(2n+1)}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

11.

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \left(1 - q^{2(n-k)}\right) \left(1 + q^{2(nm+k)}\right) q^{k^2-2kn} (-1)^k$$
$$= (-1)^n q^{-n^2-1} \frac{(1+q) \left(1-q^{2n}\right)^2 \left(1 - q^{2(m+1)n}\right)}{(1+q^{2n-1})} \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

12.

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \left(1 - q^{2n+1-k}\right)^2 \left(1 + q^{(2n+1)m+k}\right)^2 q^{k^2-2kn-k} (-1)^k$$
$$= 2 (-1)^n q^{-n^2} \left(1 - q^{2n+1}\right)^2 \left(1 + q^{m(2n+1)}\right) \left(1 + q^{(m+1)(2n+1)-1}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

13.

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \left(1 - q^{2n-k}\right)^2 \left(1 + q^{2nm+k}\right)^2 q^{k^2-2kn} (-1)^k$$
$$= (-1)^n q^{-n^2} \frac{\left(1-q^{2n}\right)^3 \left(1 - q^{(2m+1)2n-1}\right)}{1+q^{2n-1}} \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

14.

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \left(1 - q^{2n+1-k}\right)^2 \left(1 + q^{m(2n+1)+k}\right)^2 \left(1 - q^k\right)^2 q^{k^2-2kn-2k} (-1)^k$$
$$= (-1)^{n+1} q^{-(n+1)^2} \frac{\left(1-q^{2n+1}\right)^2 \left(1-q^{2n}\right)^3 \left(1 - q^{(2m+1)(2n+1)}\right)}{1+q^{2n-1}} \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

15.
$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \left(1 + q^{2(nm+k)}\right) q^{k^2-2kn-k} (-1)^k = 2 (-1)^n q^{-n^2} \left(1 + q^{2(m+1)n}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

16.
$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \left(1 - q^{2n+k}\right)^2 q^{k^2-2kn-k} (-1)^k = 2 (-1)^n q^{-n^2} \left(1 - q^{4n}\right) \left(1 - q^{4n-2}\right) \begin{bmatrix} 2n-2 \\ n-1 \end{bmatrix}_{q^2}.$$

17.
$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \left(1 - q^{2n+k}\right) \left(1 - q^k\right) q^{k^2-2kn-k} (-1)^k = (-1)^n q^{-n^2} \left(1 - q^{2n}\right)^2 \begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix}_{q^2}.$$

18.

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \left(1 - q^{2(n+k)+3}\right) \left(1 - q^{2(n+k)+1}\right) q^{k^2-2kn-3k} (-1)^k$$
$$= 2 (-1)^{n+1} q^{-n^2-2n-2} (1+q) \left(1 - q^{8n+6}\right) \left(1 - q^{2n+1}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

19.

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \left(1-q^{2(n+k)+1}\right)\left(1-q^{2k}\right) q^{k^2-2kn-3k} (-1)^k$$
$$= 2(-1)^{n+1} q^{-n^2-2n-2} (1+q) \left(1-q^{3(2n+1)}\right)\left(1-q^{2n+1}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

20.

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \left(1-q^{2n-k+1}\right)\left(1-q^{2n+k+1}\right) q^{k^2-2kn-k} (-1)^k$$
$$= (-1)^n q^{-n^2} \left(1-q^{2n+1}\right)^2 \left(1+q^{2n}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

21.

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \left(1-q^{2(n-k)+1}\right)\left(1-q^{2(n+k)+1}\right) q^{k^2-2kn-k} (-1)^k$$
$$= 2(-1)^n q^{-n^2-1} (1+q) \left(1-q^{2n+1}\right)\left(1-q^{4n+2}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

22.

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \left(1-q^{2(n-k)}\right)\left(1-q^{2(n+k)}\right) q^{k^2-2kn} (-1)^k$$
$$= (-1)^n q^{-n^2-1} \frac{\left(1-q^{2n}\right)\left(1-q^{8n}\right)(1+q)}{\left(1+q^{2n-1}\right)\left(1+q^{2n}\right)} \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

23.

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \left(1-q^{2n-k}\right)^2 \left(1-q^{2n+k}\right)^2 q^{k^2-2kn} (-1)^k$$
$$= (-1)^n q^{-n^2} \frac{\left(1-q^{2n}\right)^3\left(1-q^{6n-1}\right)}{\left(1+q^{2n-1}\right)} \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

24.

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \left(1-q^{2n-k+1}\right)^2 \left(1-q^{2n+k+1}\right)^2 \left(1-q^k\right)^2 q^{k^2-2kn-2k} (-1)^k$$
$$= (-1)^{n+1} q^{-(n+1)^2} \frac{\left(1-q^{3(2n+1)}\right)\left(1-q^{2n+1}\right)^2\left(1-q^{2n}\right)^3}{\left(1+q^{2n-1}\right)} \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

25.

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \left(1-q^{2n-k+1}\right)^2 \left(1-q^{2n+k+1}\right)^2 \left(1-q^{2k}\right)^2 q^{k^2-2kn-3k} (-1)^k$$
$$= 2(-1)^{n+1} q^{-n^2-2n-3} \frac{\left(1-q^3\right)\left(1+q^{4n+2}\right)\left(1-q^{2n+1}\right)^3\left(1-q^{2n}\right)^2\left(1-q^{4n}\right)}{\left(1+q^{2n-2}\right)\left(1+q^{2n-1}\right)(1-q)} \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

26.

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \left(1-q^{2n-k+1}\right)\left(1-q^{2n+k+1}\right)\left(1-q^{k}\right)\left(1-q^{3k}\right) q^{k^2-2kn-3k} (-1)^k$$
$$= (-1)^{n+1} q^{-n^2-2n-3} \frac{\left(1-q^{6n+3}\right)\left(1-q^{2n+1}\right)^2 \left(1-q^{2n}\right)(1+q)^2}{\left(1+q^{2n-1}\right)} \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

27.

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \left(1-q^{2n-3k}\right)\left(1-q^{2n+k}\right) q^{k-2kn+k^2} (-1)^k$$
$$= (-1)^n q^{-n^2-1} \frac{\left(1-q^{2n}\right)^3 \left(1-q^{2n+1}\right)}{\left(1+q^{2n-1}\right)} \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

28.

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \left(1+q^{2n-k}\right)\left(1+q^{2n+k}\right) q^{k^2-2kn} (-1)^k = (-1)^n q^{-n^2} \left(1+q^{2n}\right)^3 \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

29.

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \left(1-q^{2n-k}\right)^2 \left(1+q^{2n+k}\right)^2 \left(1-q^{k}\right)^2 q^{k^2-2kn-k} (-1)^k$$
$$= 2\,(-1)^n q^{-n^2-1} \frac{\left(1-q^{2n}\right)^3 \left(1-q^{2n-1}\right)\left(1+q^{2n+1}\right)\left(1+q^{4n-1}\right)}{\left(1+q^{2n-1}\right)\left(1+q^{2n-2}\right)} \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

30.

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \left(1-q^{2n+1-k}\right)\left(1+q^{2n+k+1}\right)\left(1-q^{k}\right)\left(1+q^{3k}\right) q^{k^2-2kn-3k} (-1)^k$$
$$= (-1)^{n+1} q^{-n^2-2n-3} \frac{\left(1-q^{2n+1}\right)^2 \left(1-q^{2n}\right)\left(1-q^{6n+3}\right)(1+q)^2}{\left(1+q^{2n-1}\right)} \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.$$

4 Proofs

We only choose first identity to prove as showcase. Consider

$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U^2 U_{2k} V_{m(2n+1)+2k} (-1)^k = -2U_2 U_{2n+1} V_{(m+2)(2n+1)} \begin{Bmatrix} 2n-1 \\ n \end{Bmatrix}_{U;2}.$$

First we convert the left-hand side of the claim in q-notation:

$$\sum_{k=0}^{2n+1} \begin{Bmatrix} 2n+1 \\ k \end{Bmatrix}_U^2 U_{2k} V_{m(2n+1)+2k} (-1)^k$$
$$= \frac{(-1)^{mn}\, \mathbf{i}^{m-1} q^{-\frac{1}{2}(m+2mn-1)}}{(1-q)}$$
$$\times \sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 (-1)^k q^{k^2-2kn-3k} \left(1-q^{2k}+q^{2k+m(2n+1)}-q^{4k+m(2n+1)}\right).$$

Second we convert the right-hand side of the claim in q-notation:

$$
\begin{aligned}
&-2U_2U_{2n+1}V_{(m+2)(2n+1)}\left\{\begin{matrix}2n-1\\ n\end{matrix}\right\}_{U;2}\\
&=-2\,(-1)^{n(m+1)}\,\mathbf{i}^{m-1}q^{-\frac{1}{2}(m+3)-\left(2n+mn+n^2\right)}\\
&\times\frac{\left(1-q^{2n+1}\right)}{(1-q)}\,(1+q)\left(1+q^{(m+2)(2n+1)}\right)\begin{bmatrix}2n-1\\ n\end{bmatrix}_{q^2}
\end{aligned}
$$

Thus we need to prove that

$$
\begin{aligned}
&\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\ k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn-3k}\left(1-q^{2k}+q^{2k+m(2n+1)}-q^{4k+m(2n+1)}\right)\\
&\qquad=2\,(-1)^{n+1}\,q^{-\left(n^2+2n+2\right)}\left(1-q^{2n+1}\right)(1+q)\left(1+q^{(m+2)(2n+1)}\right)\begin{bmatrix}2n-1\\ n\end{bmatrix}_{q^2}.
\end{aligned}
$$

Let

$$
S_1=\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\ k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn-3k},\quad S_2=\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\ k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn-k}
$$

$$
\text{and}\quad S_3=\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\ k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn+k}.
$$

Thus for S_1, consider

$$
\begin{aligned}
S_1&=\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\ k\end{bmatrix}_q^2(-1)^kq^{k^2-2kn-3k}\\
&=q^{-2n^2-n}\sum_{k=0}^{2n+1}\begin{bmatrix}2n+1\\ k\end{bmatrix}_q\begin{bmatrix}2n+1\\ 2n+1-k\end{bmatrix}_q(-1)^kq^{\binom{2n+1-k}{2}}q^{\binom{k}{2}-2k}\\
&=q^{-2n^2-n}\left[z^{2n+1}\right]\left(\sum_{k\geq 0}\begin{bmatrix}2n+1\\ k\end{bmatrix}_q(-1)^kq^{\binom{k}{2}-2k}z^k\right)\cdot\left(\sum_{k\geq 0}\begin{bmatrix}2n+1\\ k\end{bmatrix}_qq^{\binom{k}{2}}z^k\right)
\end{aligned}
$$

$$
\begin{aligned}
&=q^{-2n^2-n}\left[z^{2n+1}\right](z/q^2;q)_{2n+1}(-z;q)_{2n+1}\\
&=-q^{-2n^2-n-2}\left[z^{2n}\right](z^2;q^2)_{2n-1}(1+q)(1-q^{2n+1})\\
&\quad+q^{-2n^2+n-4}\left[z^{2n-2}\right](z^2;q^2)_{2n-1}(1+q)(1-q^{2n+1})\\
&=(1+q)(1-q^{2n+1})\Bigg((-q^{-2n^2-n-2}q^{2\binom{n}{2}}\begin{bmatrix}2n-1\\ n\end{bmatrix}_{q^2}(-1)^n\\
&\quad+q^{-2n^2+n-4}q^{2\binom{n-1}{2}}\begin{bmatrix}2n-1\\ n-1\end{bmatrix}_{q^2}(-1)^{n-1}\Bigg)\\
&=2(-1)^{n-1}q^{-n^2-2n-2}(1+q)(1-q^{2n+1})\begin{bmatrix}2n-1\\ n\end{bmatrix}_{q^2}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
S_2 &= \sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 (-1)^k q^{k^2-2kn-k} \\
&= q^{-2n^2-n} \left[z^{2n+1}\right] \left(\sum_{k\ge 0} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\binom{k}{2}} z^k \right) \cdot \left(\sum_{k\ge 0} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q q^{\binom{k}{2}} z^k \right) \\
&= q^{-2n^2-n} \left[z^{2n+1}\right] (z;q)_{2n+1}(-z;q)_{2n+1} \\
&= q^{-2n^2-n} \left[z^{2n+1}\right] (z^2;q^2)_{2n+1} = 0.
\end{aligned}
$$

A similar computation gives

$$
S_3 = 2(-1)^n q^{-n^2+2n}(1+q)(1-q^{2n+1}) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2}.
$$

Thus our sum in q-notation is

$$
\begin{aligned}
&\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 (-1)^k q^{k^2-2kn-3k} \left(1 - q^{2k} + q^{2k+m(2n+1)} - q^{4k+m(2n+1)}\right) \\
&\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 (-1)^k q^{k^2-2kn-3k} \left(1 - q^{2k}\left(1 - q^{m(2n+1)}\right) - q^{4k+m(2n+1)}\right) \\
&= S_1 - \left(1 - q^{m(2n+1)}\right) S_2 - q^{m(2n+1)} S_3 \\
&= 2\,(-1)^{n+1} q^{-\left(n^2+2n+2\right)} \left(1 - q^{2n+1}\right)(1+q)\left(1 + q^{(m+2)(2n+1)}\right) \begin{bmatrix} 2n-1 \\ n \end{bmatrix}_{q^2},
\end{aligned}
$$

as claimed.

We leave the other identities that could be done similarly.

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