EVALUATION OF SUMS CONTAINING TRIPLE AERATED GENERALIZED FIBONOMIAL COEFFICIENTS

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ABSTRACT. We evaluate a class of sums of triple aerated Fibonomial coefficients with a generalized Fibonacci number as coefficient. The technique is to rewrite everything in terms of a variable q and then to use Rothe's identity from classical q-calculus.

1. INTRODUCTION

Define the second order linear sequences $\{U_n\}$ and $\{V_n\}$ for $n \geq 2$ by

$$U_n = pU_{n-1} + U_{n-2}, \quad U_0 = 0,\ U_1 = 1,$$
$$V_n = pV_{n-1} + V_{n-2}, \quad V_0 = 2,\ \ V_1 = p.$$

For $n \geq k \geq 1$ and an integer m, define the generalized Fibonomial coefficient with indices in an arithmetic progression by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U;m} := \frac{U_m U_{2m} \dots U_{nm}}{(U_m U_{2m} \dots U_{km})(U_m U_{2m} \dots U_{(n-k)m})}$$

with $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\}_{U;m} = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\}_{U;m} = 1$. When $p = m = 1$, we obtain the usual Fibonomial coefficients, denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_F$. When $m = 1$, we obtain the generalized Fibonomial coefficients, denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{U;1}$. We will frequently denote $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{U;1}$ by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_U$.

As an interesting generalization of the binomial coefficients, the Fibonomial coefficients have taken the interest of several authors (for more details, see [2, 3, 4, 9]).

In a recent paper, Marques and Trojovsky [8] computed various sums of the Fibonomial coefficients with the Fibonacci and Lucas numbers as coefficients. For example, for positive integers m and n, they showed that

$$\sum_{j=0}^{4m+2} (-1)^{\frac{j(j-1)}{2}} \left\{ \begin{matrix} 4m \\ j \end{matrix} \right\}_F L_{2m-j} = -\left\{ \begin{matrix} 4m \\ 4n+1 \end{matrix} \right\}_F \frac{F_{4n+1}}{F_{2m}},$$

$$\sum_{j=0}^{4m+2} (-1)^{\frac{j(j+1)}{2}} \left\{ \begin{matrix} 4m+2 \\ j \end{matrix} \right\}_F L_{2m+1-j} = -\left\{ \begin{matrix} 4m+2 \\ 4n+3 \end{matrix} \right\}_F \frac{F_{4n+3}}{F_{2m+1}}$$

and

$$\sum_{j=0}^{4m+2} (-1)^{\frac{j(j-1)}{2}} \left\{ \begin{matrix} 4m \\ j \end{matrix} \right\}_F F_{n+4m-j} = \frac{1}{2} F_{2m+n} \sum_{j=0}^{4m} (-1)^{\frac{j(j-1)}{2}} \left\{ \begin{matrix} 4m \\ j \end{matrix} \right\}_F L_{2m-j}.$$

The authors of [6, 7] computed some generalized Fibonomial sums with the generalized Fibonacci and Lucas numbers as coefficients. For nonnegative integers n and m, they showed that

$$\sum_{k=0}^{2n+1} \left\{ \begin{matrix} 2n+1 \\ k \end{matrix} \right\}_U V_{2mk} = P_{n,m} \sum_{k=0}^{m} \left\{ \begin{matrix} 2m \\ 2k \end{matrix} \right\}_U V_{(2n+1)2k},$$

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$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\}_U U_{(2m-1)k} = P_{n,m} \sum_{k=1}^{m} \left\{ {2m-1 \atop 2k-1} \right\}_U U_{(4k-2)n},$$

where

$$P_{n,m} = \begin{cases} \prod_{k=0}^{n-m} V_{2k} & \text{if } n \geq m, \\ \prod_{k=1}^{m-n-1} V_{2k}^{-1} & \text{if } n < m. \end{cases}$$

As double aerated generalized Fibonomial sums in arithmetic progressions, we recall the following results from [6, 7]: For any positive integers n and m,

$$\sum_{k=0}^{n} (-1)^k \left\{ {2n+1 \atop 2k} \right\}_{U;m} = (-1)^{\binom{n+1}{2}} \begin{cases} \sum_{k=1}^{n} V_{mk}^2 & \text{if } m \text{ is odd,} \\ \sum_{k=1}^{n} V_{2mk} & \text{if } m \text{ is even} \end{cases}$$

and

$$\sum_{k=0}^{n} (-1)^k \left\{ {2n+1 \atop 2k+1} \right\}_{U;m} = (-1)^{\binom{n}{2}} \begin{cases} \sum_{k=1}^{n} V_{mk}^2 & \text{if } m \text{ is odd,} \\ \sum_{k=1}^{n} V_{2mk} & \text{if } m \text{ is even.} \end{cases}$$

Recently, Kılıç and Prodinger [5] gave a systematic approach to compute certain sums of squares of Fibonomial coefficients with finite products of the generalized Fibonacci and Lucas numbers as coefficients. For example, if n is a nonnegative integer and r is an arbitrary integer, then

$$\sum_{k=0}^{2n+1} \left\{ {2n+1 \atop k} \right\}_U^2 U_{k+r}^2 = U_{2n+1} U_{2n+1+2r} \left\{ {2n \atop n} \right\}_{U;2}$$

and

$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\}_U^2 U_k^4 = U_{2n-1} U_{2n}^2 U_{2n+1} \left\{ {2n-2 \atop n-1} \right\}_{U;2}.$$

As binomial sums, there exist not-so-famous double aerated binomial sums given by

$$\sum_{k=0}^{\infty} \binom{n}{2k} (-3)^k = \begin{cases} (-2)^n & \text{if } n \equiv 0 \pmod 3, \\ (-2)^{n-1} & \text{if } n \equiv 1 \pmod 3, \\ (-2)^{n-1} & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

and

$$\sum_{k=0}^{\infty} \binom{n}{2k+1} (-3)^k = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3, \\ (-2)^{n-1} & \text{if } n \equiv 1 \pmod 3, \\ (-1)^n 2^{n-1} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

In this paper, motivated by double aerated generalized Fibonomial and binomial sums mentioned above, we will compute the triple aerated generalized Fibonomial sums of the form

$$\sum_{k=0}^{cn+\lambda} \left\{ {cn+\lambda \atop 3k+\delta} \right\}_U U_{3\mu k+\gamma} (-1)^{\binom{k}{2}},$$

where c is a nonnegative integer, μ and γ are arbitrary integers, λ and δ are integers such that $0 \leq \lambda \leq 3$, $0 \leq \delta \leq 2$.

Our approach is as follows. We use the Binet forms

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{1-q^n}{1-q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n (1+q^n)$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$ where $\alpha, \beta = (p \pm \sqrt{\Delta})/2$ and $\Delta = p^2 + 4$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_{U;m} = \alpha^{mk(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^m} \quad \text{with} \quad q = -\alpha^{-2}.$$

We recall that one version of the *Cauchy binomial theorem* is given by

$$\sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = \prod_{k=1}^{n} (1 + xq^k),$$

and *Rothe's* formula [1] is

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (x;q)_n = \prod_{k=0}^{n-1} (1 - xq^k).$$

All the identities we will derive hold for general q, and results about generalized Fibonacci and Lucas numbers come out as corollaries for the special choice of q.

2. Triple Aerated Fibonomial Sums

As we mentioned before, we compute the generalized Fibonomial sums of the form

$$\sum_{k=0}^{cn+\lambda} \left\{ \begin{matrix} cn+\lambda \\ 3k+\delta \end{matrix} \right\}_U U_{3\mu k+\gamma} (-1)^{\binom{k}{2}},$$

where c is a nonnegative integer, μ and γ are arbitrary integers, λ and δ are integers such that $0 \le \lambda \le 3$ and $0 \le \delta \le 2$.

First we note that our experiments show that the parameter c must be 4. After that we thus take $c = 4$, that is, we consider the sums of the form

$$\sum_{k=0}^{4n+\lambda} \left\{ \begin{matrix} 4n+\lambda \\ 3k+\delta \end{matrix} \right\}_U U_{3\mu k+\gamma} (-1)^{\binom{k}{2}}.$$

In order to compute the claimed generalized Fibonomial sums, first we convert them into q-form and then compute it by q-analysis and the Rothe identity from classical q-calculus. Then we convert the results in q-form to the generalized Fibonomial sums to obtain claimed generalized Fibonomial sums.

Throughout this paper we denote the roots of the equation $z^2 + z + 1 = 0$ by w and $\overline{w}$, where $\overline{w}$ is the complex conjugate of w.

Now we convert the generalized Fibonomial sums into q-form :

$$\begin{aligned} &\sum_{k=0}^{4n+\lambda} \left\{ \begin{matrix} 4n+\lambda \\ 3k+\delta \end{matrix} \right\}_U U_{3\mu k+\gamma} (-1)^{\binom{k}{2}} \\ &= \alpha^{\gamma+\lambda\delta+4\delta n-\delta^2-1} \sum_{k=0}^{4n+\lambda} \begin{bmatrix} 4n+\lambda \\ 3k+\delta \end{bmatrix}_q \alpha^{3\mu k+3\lambda k-6k\delta+12kn-9k^2} \left(1 - q^{3\mu k+\gamma}\right) (-1)^{k(k-1)/2}. \\ &= \alpha^{\gamma+\lambda\delta+4\delta n-\delta^2-1} \sum_{k=0}^{4n+\lambda} \begin{bmatrix} 4n+\lambda \\ 3k+\delta \end{bmatrix}_q \alpha^{3\mu k+3\lambda k-6k\delta+12kn-9k^2} \left(1 - q^{3\mu k+\gamma}\right) (-1)^{k(k-1)/2} \end{aligned}$$

By ignoring the constant factor, we are interested in to compute the sum

$$\sum_{k=0}^{4n+\lambda} \begin{bmatrix} 4n+\lambda \\ 3k+\delta \end{bmatrix}_q q^{-\frac{3}{2}k(\mu+\lambda-3k-2\delta+4n)} \left(1 - q^{3\mu k+\gamma}\right) (-1)^{\frac{1}{2}(3\mu+3\lambda-1)k+k\delta}.$$

In order to compute the claimed sums in closed form by our approach, we have to have the sign $(-1)^k$. For this, we consider three cases of δ. Before starting to examine the cases, we will note the following result for further use: For any function f of k, we have that

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ 3k \end{bmatrix}_q f(k) = \frac{1}{3} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q f\left(\frac{k}{3}\right) \left(1 + w^k + \overline{w}^k\right), \tag{1}$$

where w is defined as before.

(1) First we start with the case $\delta = 0$. In that case, in order to have the sign $(-1)^k$, $3\mu + 3\lambda - 1$ must be an even integer of the form $2t$ such that t is an odd integer. Now we examine the following four subcases:

i) For $\lambda = 0$, we obtain the equation $3\mu = 2t + 1$, where t is an odd integer. Here we see that any solution μ should be form $4p + 1$ for $p \geq 0$. Thus by using (1), we will compute the sums

$$\sum_{k=0}^{4n} \begin{bmatrix} 4n \\ 3k \end{bmatrix}_q q^{-\frac{3}{2}k(4p+1-3k+4n)} \left(1 - q^{3(4p+1)k+\gamma}\right) (-1)^k$$
$$= \frac{1}{3} \sum_{k=0}^{4n} \begin{bmatrix} 4n \\ k \end{bmatrix}_q q^{\binom{k}{2}-2k(n+p)} \left(1 - q^{(4p+1)k+\gamma}\right) (-1)^k \left(1 + w^k + \overline{w}^k\right)$$

for $0 \leq p \leq n - 1$.

ii) For $\lambda = 1$, we obtain the equation $3\mu = 2t - 2$. The equation has a unique solution, namely $\mu = 0$ for only $t = 1$. Thus we will compute the sums

$$(1 - q^{\gamma}) \sum_{k=0}^{4n+1} \begin{bmatrix} 4n+1 \\ 3k \end{bmatrix}_q q^{-\frac{3}{2}k(4n-3k+1)} (-1)^k$$

or ignoring the constant factor and by (1), we will consider

$$\sum_{k=0}^{4n+1} \begin{bmatrix} 4n+1 \\ 3k \end{bmatrix}_q q^{-\frac{3}{2}k(4n-3k+1)} (-1)^k = \frac{1}{3} \sum_{k=0}^{4n+1} \begin{bmatrix} 4n+1 \\ k \end{bmatrix}_q q^{\binom{k}{2}-2kn} (-1)^k \left(1 + w^k + \overline{w}^k\right).$$

iii) For $\lambda = 2$, we obtain the equation $3\mu = 2t - 5$. For odd t, clearly any solution μ has the form $4p - 1$ for $p > 0$. Thus by using (1), we will compute the sums

$$\sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2 \\ 3k \end{bmatrix}_q q^{\frac{3}{2}k(3k-4n-4p-1)} \left(1 - q^{3(4p-1)k+\gamma}\right) (-1)^k$$
$$= \frac{1}{3} \sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2 \\ k \end{bmatrix}_q q^{\binom{k}{2}-2k(n+p)} \left(1 - q^{(4p-1)k+\gamma}\right) (-1)^k \left(1 + w^k + \overline{w}^k\right)$$

for $0 < p \leq n - 1$.

iv) For $\lambda = 3$, we obtain the equation $3\mu = 2t - 8$. But the equation has no integer solution μ for any odd integer t. Thus there is no closed form for the sums.

2. As a second main case, we consider the case $\delta = 1$. Then we obtain the equation $3\mu + 3\lambda - 1 = 4t$ for all t. In that case, by (1), we will compute the sums

$$\sum_{k=0}^{4n+\lambda} \begin{bmatrix} 4n+\lambda \\ 3k+1 \end{bmatrix}_q q^{-\frac{3}{2}k(\mu+\lambda-3k+4n-2)} \left(1 - q^{3\mu k+\gamma}\right) (-1)^k$$
$$= \sum_{k=0}^{4n+\lambda} \begin{bmatrix} 4n+\lambda \\ k+1 \end{bmatrix}_q q^{-\frac{1}{2}k(\mu+\lambda-k+4n-2)} \left(1 - q^{\mu k+\gamma}\right) (-1)^k \left(1 + w^k + \overline{w}^k\right).$$

In this case we have the following four subcases:

i) For $\lambda = 0$, we obtain the equation $3\mu = 4t + 1$. For all t, clearly any solution μ has the form $4p + 3$ for $p \geq 0$. Thus for $0 \leq p \leq n - 1$, we will compute the sums

$$\sum_{k=0}^{4n} \begin{bmatrix} 4n \\ k+1 \end{bmatrix}_q q^{\binom{k}{2} - 2k(n+p)} \left(1 - q^{(4p+3)k+\gamma}\right) (-1)^k .$$

ii) For $\lambda = 1$, we obtain the equation $3\mu = 4t - 2$. But the equation has no integer solution μ and so we have no closed form for the sums.

iii) For $\lambda = 2$, we obtain the equation $3\mu = 4t - 5$ for all t. In that case any solution μ has the form $4p + 1$ for $p \geq 0$. Thus we will compute the sum

$$\sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2 \\ k+1 \end{bmatrix}_q q^{\binom{k}{2} - 2k(n+p)} \left(1 - q^{(4p+1)k+\gamma}\right) (-1)^k \left(1 + w^k + \overline{w}^k\right)$$

for $0 \leq p \leq n$.

iv) For $\lambda = 3$, we obtain the equation $3\mu = 4t - 8$. But the equation has no integer solution μ and so we have no closed form for the sums.

3. As the last case, we consider the case $\delta = 2$. To have the sign $(-1)^k$, $3\mu + 3\lambda - 1$ must be an even integer of the form $2t$ such that t is an odd integer. Now we should examine the following four subcases:

i) For $\lambda = 0$, we obtain the equation $3\mu = 2t + 1$, where t is an odd integer. We see that any solution μ has the form $4p + 1$ for $p \geq 0$. Thus for $0 \leq p \leq n - 1$, by (1) we will compute the sums

$$\sum_{k=0}^{4n} \begin{bmatrix} 4n \\ 3k+2 \end{bmatrix}_q q^{-\frac{3}{2}k(4p-3k-3+4n)} \left(1 - q^{3(4p+1)k+\gamma}\right) (-1)^k$$
$$= \frac{1}{3} \sum_{k=0}^{4n} \begin{bmatrix} 4n \\ k+2 \end{bmatrix}_q q^{\binom{k}{2} - 2k(n+p-1)} \left(1 - q^{(4p+1)k+\gamma}\right) (-1)^k \left(1 + w^k + \overline{w}^k\right).$$

ii) For $\lambda = 1$, we obtain the equation $3\mu = 2t - 2$. The equation has a unique solution $\mu = 0$ for only $t = 1$. Thus we will compute the sum

$$(1 - q^\gamma) \sum_{k=0}^{4n+1} \begin{bmatrix} 4n+1 \\ 3k+2 \end{bmatrix}_q q^{-\frac{3}{2}k(4n-3k-3)} (-1)^k$$

or without constant factor and by using (1), we compute the sums

$$\sum_{k=0}^{4n+1} \begin{bmatrix} 4n+1 \\ 3k+2 \end{bmatrix}_q q^{-\frac{3}{2}k(4n-3k-3)} (-1)^k = \frac{1}{3} \sum_{k=0}^{4n+1} \begin{bmatrix} 4n+1 \\ k+2 \end{bmatrix}_q q^{\binom{k}{2} - 2k(n-1)} (-1)^k \left(1 + w^k + \overline{w}^k\right).$$

iii) For $\lambda = 2$, we obtain the equation $3\mu = 2t - 5$, where t is odd. Any solution μ has the form $4p - 1$ for $p \geq 1$. Thus for $0 \leq p \leq n + 1$, by (1), we will compute the sums

$$\sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2 \\ 3k+2 \end{bmatrix}_q q^{-\frac{3}{2}k(4p-3k-3+4n)} \left(1 - q^{3(4p-1)k+\gamma}\right) (-1)^k$$
$$= \frac{1}{3} \sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2 \\ k+2 \end{bmatrix}_q q^{\binom{k}{2} - 2k(n+p-1)} \left(1 - q^{(4p-1)k+\gamma}\right) (-1)^k \left(1 + w^k + \overline{w}^k\right).$$

iv) For $\lambda = 3$, we obtain the equation $3\mu = 2t - 8$. But the equation has no integer solution μ and so we have no closed form for the sums.

In the next section, we will compute the Gaussian q-binomial sums mentioned above and we give related generalized Fibonomial sums in the last section.

3. Evaluation of the Gaussian q-binomial Sums

We compute some of the Gaussian q-binomial sums mentioned in the previous section as showcase to don't bother the readers. We start with the the case (1.i).

1.i) For $0 \le p \le n-1$, consider

$$\begin{aligned}
&\sum_{k=0}^{4n} \begin{bmatrix} 4n \\ k \end{bmatrix}_q q^{\binom{k}{2}-2k(n+p)} \left(1-q^{(4p+1)k+\gamma}\right)(-1)^k \left(1+w^k+\overline{w}^k\right) \\
&= \sum_{k=0}^{4n} \begin{bmatrix} 4n \\ k \end{bmatrix}_q q^{\binom{k}{2}-2k(n+p)} (-1)^k \left(1+w^k+\overline{w}^k\right) \\
&- \frac{1}{3} q^{\gamma} \sum_{k=0}^{4n} \begin{bmatrix} 4n \\ k \end{bmatrix}_q q^{\binom{k}{2}+k(2(p-n)+1)} (-1)^k \left(1+w^k+\overline{w}^k\right) \\
&= \sum_{k=0}^{4n} \begin{bmatrix} 4n \\ k \end{bmatrix}_q q^{\binom{k}{2}-2k(n+p)} (-1)^k + \sum_{k=0}^{4n} \begin{bmatrix} 4n \\ k \end{bmatrix}_q q^{\binom{k}{2}} \left(wq^{-2(p+n)}\right)^k (-1)^k \\
&+ \sum_{k=0}^{4n} \begin{bmatrix} 4n \\ k \end{bmatrix}_q q^{\binom{k}{2}} \left(-\overline{w}q^{-2(p+n)}\right)^k - q^{\gamma} \sum_{k=0}^{4n} \begin{bmatrix} 4n \\ k \end{bmatrix}_q q^{\binom{k}{2}} q^{k(2p-2n+1)} (-1)^k \\
&- q^{\gamma} \sum_{k=0}^{4n} \begin{bmatrix} 4n \\ k \end{bmatrix}_q q^{\binom{k}{2}} \left(-wq^{(2(p-n)+1)}\right)^k - q^{\gamma} \sum_{k=0}^{4n} \begin{bmatrix} 4n \\ k \end{bmatrix}_q q^{\binom{k}{2}} \left(-\overline{w}q^{(2(p-n)+1)}\right)^k,
\end{aligned}$$

which, by Rothe's identity, equals

$$\begin{aligned}
&(q^{-2(p+n)};q)_{4n} + \left(wq^{-2(p+n)};q\right)_{4n} + \left(\overline{w}q^{-2(p+n)};q\right)_{4n} \\
&-q^{\gamma}\left(\left(q^{2(p-n)+1};q\right)_{4n} + \left(wq^{2(p-n)+1};q\right)_{4n} + \left(\overline{w}q^{2(p-n)+1};q\right)_{4n}\right).
\end{aligned} \tag{2}$$

Here if $-n \le p < n$, then

$$(q^{-2(p+n)};q)_{4n} = 0 \text{ and } \left(q^{2(p-n)+1};q\right)_{4n} = 0$$

and so the equation (2) is equal to

$$\begin{aligned}
&\left(wq^{-2(p+n)};q\right)_{4n} + \left(\overline{w}q^{-2(p+n)};q\right)_{4n} \\
&- q^{\gamma}\left(\left(wq^{2(p-n)+1};q\right)_{4n} + \left(\overline{w}q^{2(p-n)+1};q\right)_{4n}\right) \\
&= (1-w)\left(\prod_{k=0}^{2n+2p-1}\left(1-wq^{k-2p-2n}\right)\right)\left(\prod_{k=2n+2p+1}^{4n-1}\left(1-wq^{k-2p-2n}\right)\right) \\
&+ (1-\overline{w})\left(\prod_{k=0}^{2n+2p-1}\left(1-\overline{w}q^{k-2p-2n}\right)\right)\left(\prod_{k=2n+2p+1}^{4n-1}\left(1-\overline{w}q^{k-2p-2n}\right)\right) \\
&- q^{\gamma}(1-w)\left(\prod_{k=0}^{2n-2p-2}\left(1-wq^{k-(2n-2p-1)}\right)\right)\left(\prod_{k=2n-2p}^{4n-1}\left(1-wq^{k-(2n-2p-1)}\right)\right) \\
&- q^{\gamma}(1-\overline{w})\left(\prod_{k=0}^{2n-2p-2}\left(1-\overline{w}q^{k-(2n-2p-1)}\right)\right)\left(\prod_{k=2n-2p}^{4n-1}\left(1-\overline{w}q^{k-(2n-2p-1)}\right)\right),
\end{aligned}$$

which, by some rearrangements, equals

$$
\begin{aligned}
&(1-w)\left(\prod_{k=1}^{2n+2p}\left(1-wq^{-k}\right)\right)\left(\prod_{k=1}^{2n-2p-1}\left(1-wq^{k}\right)\right)\\
&+(1-\overline{w})\left(\prod_{k=1}^{2n+2p}\left(1-\overline{w}q^{-k}\right)\right)\left(\prod_{k=1}^{2n-2p-1}\left(1-\overline{w}q^{k}\right)\right)\\
&-q^{\gamma}(1-w)\left(\prod_{k=1}^{2n-2p-1}\left(1-wq^{-k}\right)\right)\left(\prod_{k=1}^{2n+2p}\left(1-wq^{k}\right)\right)\\
&-q^{\gamma}(1-\overline{w})\left(\prod_{k=1}^{2n-2p-1}\left(1-\overline{w}q^{-k}\right)\right)\left(\prod_{k=1}^{2n+2p}\left(1-\overline{w}q^{k}\right)\right).
\end{aligned}
\tag{3}
$$

For $p \geq 0$, the equation (3) is equal to

$$
\begin{aligned}
&-q^{(n-p)(2p-2n+1)}w^{-2p+2n-1}(1-w)\left(\prod_{k=2n-2p}^{2n+2p}\left(1-wq^{-k}\right)\right)\left(\prod_{k=1}^{2n-2p-1}\frac{1-q^{3k}}{1-q^{k}}\right)\\
&-q^{(n-p)(2p-2n+1)}\overline{w}^{-2p+2n-1}(1-\overline{w})\left(\prod_{k=2n-2p}^{2n+2p}\left(1-\overline{w}q^{-k}\right)\right)\left(\prod_{k=1}^{2n-2p-1}\frac{1-q^{3k}}{1-q^{k}}\right)\\
&+q^{(n-p)(2p-2n+1)}w^{-2p+2n-1}q^{\gamma}(1-w)\left(\prod_{k=2n-2p}^{2n+2p}\left(1-wq^{k}\right)\right)\left(\prod_{k=1}^{2n-2p-1}\frac{1-q^{3k}}{1-q^{k}}\right)\\
&+q^{(n-p)(2p-2n+1)}\overline{w}^{-2p+2n-1}q^{\gamma}(1-\overline{w})\left(\prod_{k=2n-2p}^{2n+2p}\left(1-\overline{w}q^{k}\right)\right)\left(\prod_{k=1}^{2n-2p-1}\frac{1-q^{3k}}{1-q^{k}}\right).
\end{aligned}
$$

Thus by combining common statements, the last equation is equal to

$$
\begin{aligned}
&q^{(n-p)(2p-2n+1)}\left(\prod_{k=1}^{2n-2p-1}\frac{1-q^{3k}}{1-q^{k}}\right)\\
&\times\Bigg(-w^{-2p+2n-1}(1-w)\prod_{k=2n-2p}^{2n+2p}\left(1-wq^{-k}\right)-\overline{w}^{-2p+2n-1}(1-\overline{w})\prod_{k=2n-2p}^{2n+2p}\left(1-\overline{w}q^{-k}\right)\\
&+w^{-2p+2n-1}q^{\gamma}(1-w)\prod_{k=2n-2p}^{2n+2p}\left(1-wq^{k}\right)+\overline{w}^{-2p+2n-1}q^{\gamma}(1-\overline{w})\prod_{k=2n-2p}^{2n+2p}\left(1-\overline{w}q^{k}\right)\Bigg).
\end{aligned}
$$

Since

$$
-\overline{w}^{4p+1}q^{(8p+2)n}\prod_{k=2n-2p}^{2n+2p}\left(1-wq^{-k}\right) = \prod_{k=2n-2p}^{2n+2p}\left(1-\overline{w}q^{k}\right),
$$

consequently we derive

$$
\begin{aligned}
&\sum_{k=0}^{4n}\begin{bmatrix}4n\\3k\end{bmatrix}_q q^{-\frac{3}{2}k(4p+1-3k+4n)}\left(1-q^{3(4p+1)k+\gamma}\right)(-1)^k\\
&=-\frac{1}{3}q^{(n-p)(2p-2n+1)}(1-w)\frac{\left(q^3;q^3\right)_{2n-2p-1}}{(q;q)_{2n-2p-1}}
\end{aligned}
$$

$$\times\left[\left(w^{-2p+2n-1}-q^{2(4p+1)n+\gamma}\overline{w}^{2p+2n+1}\right)\prod_{k=2n-2p}^{2n+2p}\left(1-wq^{-k}\right)\right.$$
$$\left.+\left(w^{2p+2n}q^{2(4p+1)n+\gamma}-\overline{w}^{2n-2p}\right)\prod_{k=2n-2p}^{2n+2p}\left(1-\overline{w}q^{-k}\right)\right].$$

As special case, for $p=0$, we have the result

$$\sum_{k=0}^{4n}\begin{bmatrix}4n\\3k\end{bmatrix}_q q^{-\frac{3k}{2}(4n-3k+1)}\left(1-q^{3k+\gamma}\right)(-1)^k=-q^{-n(2n+1)}\frac{\left(q^3;q^3\right)_{2n-1}}{(q;q)_{2n-1}}$$
$$\times\begin{cases}\left(1-q^{\gamma+2n}\right)\left(1+q^{2n}\right) & \text{if } n\equiv 0 \pmod 3,\\ \left(1-q^{\gamma+4n}\right) & \text{if } n\equiv 1 \pmod 3,\\ q^{2n}\left(1-q^{\gamma}\right) & \text{if } n\equiv 2 \pmod 3.\end{cases}$$

For $p=\gamma=0$, we have

$$\sum_{k=0}^{4n}\begin{bmatrix}4n\\3k\end{bmatrix}_q q^{-\frac{3k}{2}(4n-3k+1)}\left(1-q^{3k}\right)(-1)^k = -q^{-n(2n+1)}\frac{\left(q^3;q^3\right)_{2n-1}}{(q;q)_{2n-1}}\begin{cases}\left(1-q^{4n}\right) & \text{if } n\equiv 0,1 \pmod 3,\\ 0 & \text{if } n\equiv 2 \pmod 3.\end{cases}$$

For $p=0$ and $\gamma=1$, we have

$$\sum_{k=0}^{4n}\begin{bmatrix}4n\\3k\end{bmatrix}_q q^{-\frac{3}{2}k(4n-3k+1)}\left(1-q^{3k+1}\right)(-1)^k=-q^{-n(2n+1)}\frac{\left(q^3;q^3\right)_{2n-1}}{(q;q)_{2n-1}}$$
$$\times\begin{cases}\left(1-q^{2n+1}\right)\left(1+q^{2n}\right) & \text{if } n\equiv 0 \pmod 3,\\ \left(1-q^{4n+1}\right) & \text{if } n\equiv 1 \pmod 3,\\ q^{2n}\left(1-q\right) & \text{if } n\equiv 2 \pmod 3.\end{cases}$$

1.ii) Now without constant factor, we consider

$$\sum_{k=0}^{4n+1}\begin{bmatrix}4n+1\\k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}-2kn}\left(1+w^k+\overline{w}^k\right)$$
$$=\sum_{k=0}^{4n+1}\begin{bmatrix}4n+1\\k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}-2kn}+\sum_{k=0}^{4n+1}\begin{bmatrix}4n+1\\k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}}\left(q^{-2n}w\right)^k$$
$$+\sum_{k=0}^{4n+1}\begin{bmatrix}4n+1\\k\end{bmatrix}_q(-1)^k q^{\binom{k}{2}}\left(q^{-2n}\overline{w}\right)^k,$$

which, by Rothe's formula, equals

$$\left(q^{-2n};q\right)_{4n+1}+\left(q^{-2n}w;q\right)_{4n+1}+\left(q^{-2n}\overline{w};q\right)_{4n+1}$$
$$=\prod_{k=0}^{4n}\left(1-q^{k-2n}\right)+\prod_{k=0}^{4n}\left(1-wq^{k-2n}\right)+\prod_{k=0}^{4n}\left(1-\overline{w}q^{k-2n}\right). \tag{4}$$

Since

$$\prod_{k=0}^{4n}\left(1-wq^{k-2n}\right)=-w^{n+1}\prod_{k=0}^{4n}\left(1-\overline{w}q^{k-2n}\right)\quad\text{and}\quad\left(q^{-2n};q\right)_{4n+1}=0,$$

the equation (4) is equal to

$$\prod_{k=0}^{4n}\left(1-q^{k-2n}\right)+\left(1-\overline{w}^{n+1}\right)\prod_{k=0}^{4n}\left(1-wq^{k-2n}\right) \quad = \quad \left(1-\overline{w}^{n+1}\right)\prod_{k=0}^{4n}\left(1-wq^{k-2n}\right).$$

Thus we obtain

$$\sum_{k=0}^{4n+1}\begin{bmatrix}4n+1\\3k\end{bmatrix}q^{-\frac{1}{2}3k(4n-3k+1)}\left(-1\right)^{k}=\frac{1}{3}\left(1-w\right)\left(1-\overline{w}^{n+1}\right)w^{2n}q^{-2n^{2}-n}\frac{\left(q^{3};q^{3}\right)_{2n}}{(q;q)_{2n}}.$$

Consequently we get

$$\sum_{k=0}^{4n+1}\begin{bmatrix}4n+1\\3k\end{bmatrix}_{q}q^{-\frac{k}{2}(12n-9k+3)}\left(-1\right)^{k} \quad = \quad q^{-n(2n+1)}\frac{\left(q^{3};q^{3}\right)_{2n}}{(q;q)_{2n}}\begin{cases}1 & \text{if } n\equiv 0 \pmod 3,\\ -1 & \text{if } n\equiv 1 \pmod 3,\\ 0 & \text{if } n\equiv 2 \pmod 3.\end{cases}$$

1.iii) For $0<p\leq n-1$, we give the following result without proof

$$\begin{aligned}&\sum_{k=0}^{4n+2}\begin{bmatrix}4n+2\\3k\end{bmatrix}_{q}q^{-\frac{1}{2}k(3(4p-1)+6-9k+12n)}\left(1-q^{3(4p-1)k+\gamma}\right)\left(-1\right)^{k}\\&=\frac{1}{3}\left(1-w\right)q^{(n-p+1)(2p-2n-1)}\frac{\left(q^{3};q^{3}\right)_{2n-2p+1}}{(q;q)_{2n-2p+1}}\\&\times\Bigg[\left(w^{-2p+2n+1}q^{\gamma}-\overline{w}^{2p+2n+1}q^{-(4p-1)(2n+1)}\right)\prod_{k=2n-2p+2}^{2n+2p}\left(1-wq^{k}\right)\\&-\left(w^{-2p+2n+1}-\overline{w}^{2p+2n+1}q^{(4p-1)(2n+1)}q^{\gamma}\right)\prod_{k=2n-2p+2}^{2n+2p}\left(1-wq^{-k}\right)\Bigg].\end{aligned}$$

As a special case, for $\gamma=0$ and $p=1$, we have

$$\begin{aligned}\sum_{k=0}^{4n+2}\begin{bmatrix}4n+2\\3k\end{bmatrix}_{q}q^{-\frac{1}{2}k(15-9k+12n)}\left(1-q^{9k}\right)\left(-1\right)^{k}&=-q^{-(2n+3)(n+1)}\frac{\left(q^{3};q^{3}\right)_{2n-1}}{(q;q)_{2n-1}}\\&\times\begin{cases}\left(1-q^{4n+1}\right)\left(1-q^{4n+2}\right)\left(1-q^{4n+3}\right) & \text{if } n\equiv 0 \pmod 3,\\ q^{2n}\left(q+q^{2}+1\right)\left(1+q^{2n+1}\right)\left(1-q^{6n+3}\right) & \text{if } n\equiv 1 \pmod 3,\\ -q^{2n}\left(q+q^{2}+1\right)\left(1-q^{8n+4}\right)-\left(1-q^{12n+6}\right) & \text{if } n\equiv 2 \pmod 3.\end{cases}\end{aligned}$$

Now we give formulae for the second main case with its subcases:

2.i) For $0\leq p\leq n-1$, we have the following result without proof:

$$\begin{aligned}&\sum_{k=0}^{4n}\begin{bmatrix}4n\\3k+1\end{bmatrix}_{q}q^{-\frac{3}{2}k(4p+3-3k-2+4n)}\left(1-q^{3(4p+3)k+\gamma}\right)\left(-1\right)^{k}\\&=-\frac{1}{3}q^{2p+2n+1}q^{(2p-2n+1)(n-p-1)}\left(1-w\right)\frac{\left(q^{3};q^{3}\right)_{2(n-p-1)}}{(q;q)_{2(n-p-1)}}\\&\times\Bigg[\left(w^{2n-2p}-\overline{w}^{2n+2p-2}q^{\gamma+(4p+3)(2n-1)}\right)\prod_{k=2n-2p-1}^{2n+2p+1}\left(1-wq^{-k}\right)\\&-\left(q^{\gamma-(4p+3)}w^{2n-2p}-\overline{w}^{2n+2p-2}q^{-2n(4p+3)}\right)\prod_{k=2n-2p-1}^{2n+2p+1}\left(1-wq^{k}\right)\Bigg].\end{aligned}$$

Especially for $\gamma = p = 0$, we have the following corollary

$$\sum_{k=0}^{4n} \begin{bmatrix} 4n \\ 3k+1 \end{bmatrix}_q q^{-\frac{3}{2}k(1-3k+4n)} \left(1-q^{9k}\right)(-1)^k = q^{-n(2n+1)} \frac{\left(q^3;q^3\right)_{2(n-1)}}{(q;q)_{2(n-1)}}$$
$$\times \begin{cases} q^{2n-1}\left(1+q^{4n-2}\right)\left(1-q^{4n-1}\right)\left(\frac{1-q^3}{1-q}\right)+\left(1-q^{12n-3}\right) & \text{if } n \equiv 0 \pmod 3, \\ q^{4n-1}\left(\frac{1-q^3}{1-q}\right)\left(1-q^{4n-3}\right)-\left(1-q^{6n}\right)\left(1+q^{6n-3}\right) & \text{if } n \equiv 1 \pmod 3, \\ -q^{2n-1}\left(\frac{1-q^3}{1-q}\right)\left(1-q^{6n-3}\right)\left(1+q^{2n}\right) & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

2.ii) There is no closed formula as mentioned as before.

2.iii) For $0 \le p \le n$, consider

$$\begin{aligned}
&\sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2 \\ k+1 \end{bmatrix}_q q^{\binom{k}{2}-2k(n+p)} \left(1-q^{(4p+1)k+\gamma}\right)(-1)^k \left(1+w^k+\overline{w}^k\right) \\
&= \sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2 \\ k \end{bmatrix}_q q^{\binom{k}{2}} q^{-k(2p+2n+1)} q^{(2p+2n+1)} \left(1-q^{(4p+1)(k-1)+\gamma}\right)(-1)^{k-1} \left(1+w^{k-1}+\overline{w}^{k-1}\right) \\
&= -q^{(2p+2n+1)} \sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2 \\ k \end{bmatrix}_q q^{k(k-1)/2} q^{-k(2p+2n+1)} \left(1-q^{(4p+1)(k-1)+\gamma}\right)(-1)^k \left(1+w^{k-1}+\overline{w}^{k-1}\right).
\end{aligned}$$

Now consider the sum just above without constant factor,

$$\begin{aligned}
&\sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2 \\ k \end{bmatrix}_q q^{k(k-1)/2} q^{-k(2p+2n+1)} \left(1-q^{(4p+1)(k-1)+\gamma}\right)(-1)^k \left(1+w^{k-1}+\overline{w}^{k-1}\right) \\
&= \sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2 \\ k \end{bmatrix}_q q^{k(k-1)/2} q^{-k(2p+2n+1)} (-1)^k \left(1+w^{k-1}+\overline{w}^{k-1}\right) \\
&- q^{\gamma-(4p+1)} \sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2 \\ k \end{bmatrix}_q q^{k(k-1)/2} q^{2k(p-n)} (-1)^k \left(1+w^{k-1}+\overline{w}^{k-1}\right),
\end{aligned}$$

which, by Rothe's identity, equals

$$\begin{aligned}
&\left(q^{-(2p+2n+1)};q\right)_{4n+2} + w^{-1}\left(wq^{-(2p+2n+1)};q\right)_{4n+2} \\
&+ \overline{w}^{-1}\left(\overline{w}q^{-(2p+2n+1)};q\right)_{4n+2} - q^{\gamma-4p-1}\left(q^{-2(n-p)};q\right)_{4n+2} \\
&- q^{\gamma-4p-1}w^{-1}\left(wq^{-2(n-p)};q\right)_{4n+2} - q^{\gamma-4p-1}\overline{w}^{-1}\left(\overline{w}q^{-2(n-p)};q\right)_{4n+2} \\
&= \prod_{k=0}^{4n+1}\left(1-q^{k-(2p+2n+1)}\right) + w^{-1}\prod_{k=0}^{4n+1}\left(1-wq^{k-(2p+2n+1)}\right) \\
&+ \overline{w}^{-1}\prod_{k=0}^{4n+1}\left(1-\overline{w}q^{k-(2p+2n+1)}\right) - q^{\gamma-4p-1}\prod_{k=0}^{4n+1}\left(1-q^{k-2(n-p)}\right) \\
&- q^{\gamma-4p-1}w^{-1}\prod_{k=0}^{4n+1}\left(1-wq^{k-2(n-p)}\right) - q^{\gamma-4p-1}\overline{w}^{-1}\prod_{k=0}^{4n+1}\left(1-\overline{w}q^{k-2(n-p)}\right).
\end{aligned}$$

For $0 \le p \le n-1$, since $(q^{-(2p+2n+1)};q)_{4n+2} = (q^{-2(p+n)};q)_{4n+2} = 0$, the last equation equals

$$
\begin{aligned}
& w^{-1} \prod_{k=0}^{4n+1} \left(1 - wq^{k-(2p+2n+1)}\right) + \overline{w}^{-1} \prod_{k=0}^{4n+1} \left(1 - \overline{w}q^{k-(2p+2n+1)}\right) \\
& - q^{\gamma-(4p+1)} \left[w^{-1} \prod_{k=0}^{4n+1} \left(1 - wq^{k-2(n-p)}\right) + \overline{w}^{-1} \prod_{k=0}^{4n+1} \left(1 - \overline{w}q^{k-2(n-p)}\right) \right] \\
& = w^{-1}(1-w) \left(\prod_{k=0}^{2n+2p} \left(1 - wq^{k-(2p+2n+1)}\right) \right) \left(\prod_{k=2n+2p+2}^{4n+1} \left(1 - wq^{k-(2p+2n+1)}\right) \right) \\
& + \overline{w}^{-1}(1-\overline{w}) \left(\prod_{k=0}^{2n+2p} \left(1 - \overline{w}q^{k-(2p+2n+1)}\right) \right) \left(\prod_{k=2n+2p+2}^{4n+1} \left(1 - \overline{w}q^{k-(2p+2n+1)}\right) \right) \\
& - q^{\gamma-(4p+1)} w^{-1}(1-w) \left(\prod_{k=0}^{2n-2p-1} \left(1 - wq^{k-2(n-p)}\right) \right) \left(\prod_{k=2n-2p+1}^{4n+1} \left(1 - wq^{k-2(n-p)}\right) \right) \\
& - q^{\gamma-(4p+1)} \overline{w}^{-1}(1-\overline{w}) \left(\prod_{k=0}^{2n-2p-1} \left(1 - \overline{w}q^{k-2(n-p)}\right) \right) \left(\prod_{k=2n-2p+1}^{4n+1} \left(1 - \overline{w}q^{k-2(n-p)}\right) \right),
\end{aligned}
$$

which, by some arrangements, equals

$$
\begin{aligned}
& q^{(2p-2n-1)(n-p)} w^{2n-2p-1} (1-w) \left(\prod_{k=2n-2p+1}^{2n+2p+1} \left(1 - wq^{-k}\right) \right) \left(\prod_{k=1}^{2n-2p} \frac{1-q^{3k}}{1-q^k} \right) \\
& + q^{(2p-2n-1)(n-p)} \overline{w}^{2n-2p-1} (1-\overline{w}) \left(\prod_{k=2n-2p+1}^{2n+2p+1} \left(1 - \overline{w}q^{-k}\right) \right) \left(\prod_{k=1}^{2n-2p} \frac{1-q^{3k}}{1-q^k} \right) \\
& - q^{(2p-2n-1)(n-p)} w^{2n-2p-1} q^{\gamma-(4p+1)} (1-w) \left(\prod_{k=2n-2p+1}^{2n+2p+1} \left(1 - wq^{k}\right) \right) \left(\prod_{k=1}^{2n-2p} \frac{1-q^{3k}}{1-q^k} \right) \\
& - q^{(2p-2n-1)(n-p)} \overline{w}^{2n-2p-1} q^{\gamma-(4p+1)} (1-\overline{w}) \left(\prod_{k=2n-2p+1}^{2n+2p+1} \left(1 - \overline{w}q^{k}\right) \right) \left(\prod_{k=1}^{2n-2p} \frac{1-q^{3k}}{1-q^k} \right) \\
= \; & (1-w)\, q^{(2p-2n-1)(n-p)} \left(\prod_{k=1}^{2n-2p} \frac{1-q^{3k}}{1-q^k} \right) \\
& \times \left[w^{2n-2p-1} \prod_{k=2n-2p+1}^{2n+2p+1} \left(1 - wq^{-k}\right) - \overline{w}^{2n-2p} \prod_{k=2n-2p+1}^{2n+2p+1} \left(1 - \overline{w}q^{-k}\right) \right. \\
& \left. - q^{\gamma-(4p+1)} w^{2n-2p-1} \prod_{k=2n-2p+1}^{2n+2p+1} \left(1 - wq^{k}\right) + q^{\gamma-(4p+1)} \overline{w}^{2n-2p} \prod_{k=2n-2p+1}^{2n+2p+1} \left(1 - \overline{w}q^{k}\right) \right].
\end{aligned}
$$

Since

$$
-\overline{w}^{4j+1} q^{(4j+1)(2n+1)} \prod_{k=2n-2j+1}^{2n+2j+1} \left(1 - wq^{-k}\right) = \prod_{k=2n-2j+1}^{2n+2j+1} \left(1 - \overline{w}q^{k}\right),
$$

$$-\overline{w}^{4j+1}q^{-(4j+1)(2n+1)}\prod_{k=2n-2j+1}^{2n+2j+1}\left(1-wq^{k}\right)=\prod_{k=2n-2j+1}^{2n+2j+1}\left(1-\overline{w}q^{-k}\right),$$

consequently we get

$$\sum_{k=0}^{4n+2}\begin{bmatrix}4n+2\\3k+1\end{bmatrix}_q q^{-\frac{3}{2}k(4p+1-3k+4n)}\left(1-q^{3(4p+1)k+\gamma}\right)(-1)^k$$
$$=-\frac{1}{3}q^{2p+2n+1}q^{(2p-2n-1)(n-p)}(1-w)\frac{\left(q^3;q^3\right)_{2(n-p)}}{(q;q)_{2(n-p)}}$$
$$\times\left[\left(w^{2n-2p-1}-\overline{w}^{2n+2p+1}q^{2n(4p+1)+\gamma}\right)\prod_{k=2n-2p+1}^{2n+2p+1}\left(1-wq^{-k}\right)\right.$$
$$\left.+\left(\overline{w}^{2n+2p+1}q^{-(4p+1)(2n+1)}-q^{\gamma-(4p+1)}w^{2n-2p-1}\right)\prod_{k=2n-2p+1}^{2n+2p+1}\left(1-wq^{k}\right)\right].$$

For $\gamma=p=0$, we have the following corollary

$$\sum_{k=0}^{4n+2}\begin{bmatrix}4n+2\\3k+1\end{bmatrix}_q q^{-\frac{3}{2}k(4n-3k+1)}\left(1-q^{3k}\right)(-1)^k=q^{-n(2n+1)}\frac{\left(q^3;q^3\right)_{2n}}{(q;q)_{2n}}$$
$$\times\begin{cases}\left(1+q^{2n+1}\right)\left(1-q^{2n}\right) & \text{if } n\equiv 0 \pmod 3,\\ -\left(1-q^{4n+1}\right) & \text{if } n\equiv 1 \pmod 3,\\ (1-q)\,q^{2n} & \text{if } n\equiv 2 \pmod 3.\end{cases}$$

2.iv) For the case, there is no closed formula as mentioned as before.

Similar to the above results, we give the following results without proof.

3.i) For $0\le p\le n$,

$$\sum_{k=0}^{4n}\begin{bmatrix}4n\\3k+2\end{bmatrix}_q q^{-\frac{3}{2}k(4p+1-3k-4+4n)}\left(1-q^{3(4p+1)k+\gamma}\right)(-1)^k$$
$$=-\frac{1}{3}q^{4p+4n-1}q^{(n-p)(2p-2n+1)}\frac{\left(q^3;q^3\right)_{2n-2p-1}}{(q;q)_{2n-2p-1}}$$
$$\times\left[\left(\overline{w}^{n-p}(1-w)+w^{n+p-1}(1-\overline{w})\,q^{2(n-1)(4p+1)+\gamma}\right)\prod_{k=2n-2p}^{2n+2p}\left(1-wq^{-k}\right)\right.$$
$$\left.-\left(q^{\gamma-2(4p+1)}\overline{w}^{n-p}(1-w)+w^{n+p-1}(1-\overline{w})\,q^{-2n(4p+1)}\right)\prod_{k=2n-2p}^{2n+2p}\left(1-wq^{k}\right)\right].$$

Especially for $\gamma=p=0$, we get

$$\sum_{k=0}^{4n}\begin{bmatrix}4n\\3k+2\end{bmatrix}_q q^{-\frac{3}{2}k(-3k-3+4n)}\left(1-q^{3k}\right)(-1)^k$$
$$=q^{(2n-1)(1-n)}\frac{\left(q^3;q^3\right)_{2n-1}}{(q;q)_{2n-1}}\begin{cases}q^{2n-2}\left(1-q^2\right) & \text{if } n\equiv 0 \pmod 3,\\ \left(1-q^{2n-2}\right)\left(1+q^{2n}\right) & \text{if } n\equiv 1 \pmod 3,\\ -\left(1-q^{4n-2}\right) & \text{if } n\equiv 2 \pmod 3.\end{cases}$$

3.ii)

$$\sum_{k=0}^{4n+1} \begin{bmatrix} 4n+1 \\ 3k+2 \end{bmatrix}_q q^{-\frac{3}{2}k(-3k-3+4n)} (-1)^k = q^{-(2n-1)(n-1)} \frac{(q^3;q^3)_{2n}}{(q;q)_{2n}} \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3, \\ 1 & \text{if } n \equiv 1 \pmod 3, \\ -1 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

3.iii) For $0 < p \leq n+1$,

$$\sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2 \\ 3k+2 \end{bmatrix}_q q^{-\frac{3}{2}k(4p-1+2-3k-4+4n)} \left(1 - q^{3(4p-1)k+\gamma}\right) (-1)^k$$

$$= -\frac{1}{3} q^{4n+4p-1} q^{-(2n-2p+1)(n-p+1)} \frac{(q^3;q^3)_{2n-2p+1}}{(q;q)_{2n-2p+1}}$$

$$\times \left[\left(w^{2n-2p-1}(1-w) + q^{\gamma+(4p-1)(2n-1)} \overline{w}^{2n+2p-2}(1-\overline{w}) \right) \prod_{k=2n-2p+2}^{2n+2p} (1 - wq^{-k}) \right.$$

$$\left. - \left(q^{\gamma-8p+2} w^{2n-2p-1}(1-w) + q^{-(4p-1)(2n+1)} \overline{w}^{2n+2p-2}(1-\overline{w}) \right) \prod_{k=2n-2p+2}^{2n+2p} (1 - wq^{k}) \right].$$

Especially for $\gamma = 0$ and $p = 1$, we get

$$\sum_{k=0}^{4n+2} \begin{bmatrix} 4n+2 \\ 3k+2 \end{bmatrix}_q q^{-\frac{3}{2}k(-3k+4n)} \left(1 - q^{9k}\right) (-1)^k$$

$$= -q^{-n(2n+1)} \left(1 - q^{2n}\right) \left(1 - q^{2n+1}\right) \left(1 - q^{2n+2}\right) \left(1 + q^{6n-3}\right)$$

$$\times \frac{(q^3;q^3)_{2n-1}}{(q;q)_{2n-1}} \begin{cases} -1 & \text{if } n \equiv 0 \pmod 3, \\ 1 & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

3.iv) There is no closed formula as mentioned as before.

4. Triple aerated Generalized Fibonomial Sums

As corollaries of our results, we present sums formulae including generalized Fibonomial coefficients. From (1.i), we derive the generalized Fibonomial-Fibonacci-Lucas sums:

1.

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ 3k \end{Bmatrix}_U U_{3k} (-1)^{\binom{k}{2}} = (-1)^{n+1} \left(\prod_{t=1}^{2n-1} \frac{U_{3t}}{U_t} \right) \begin{cases} U_{4n} & \text{if } n \equiv 0,1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

2.

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ 3k \end{Bmatrix}_U U_{3k+1} (-1)^{\binom{k}{2}} = (-1)^{n+1} \left(\prod_{t=1}^{2n-1} \frac{U_{3t}}{U_t} \right) \begin{cases} V_{2n}U_{2n+1} & \text{if } n \equiv 0 \pmod 3, \\ U_{4n+1} & \text{if } n \equiv 1 \pmod 3, \\ 1 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

From (1.ii), we derive the generalized Fibonomial-Fibonacci sum :

$$\sum_{k=0}^{4n+1} \begin{Bmatrix} 4n+1 \\ 3k \end{Bmatrix}_U (-1)^{\binom{k}{2}} = (-1)^{n} \left(\prod_{t=1}^{2n} \frac{U_{3t}}{U_t} \right) \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3, \\ -1 & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

From (1.iii), we derive the generalized Fibonomial-Fibonacci-Lucas sums :

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ 3k \end{Bmatrix}_U U_{9k} (-1)^{\binom{k}{2}} = (-1)^{n+1} \left(\prod_{t=1}^{2n-1} \frac{U_{3t}}{U_t} \right) \begin{cases} -\Delta U_{4n+1} U_{4n+2} U_{4n+3} & \text{if } n \equiv 0 \pmod 3, \\ -U_3 U_{6n+3} V_{2n+1} & \text{if } n \equiv 1 \pmod 3, \\ U_3 U_{8n+4} + U_{12n+6} & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

where Δ is defined as before.

From (2.i), we derive the generalized Fibonomial-Fibonacci-Lucas sum :

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ 3k+1 \end{Bmatrix}_U U_{9k} (-1)^{\frac{1}{2}k(k-1)} = (-1)^{n+1} \left(\prod_{t=1}^{2n-2} \frac{U_{3t}}{U_t} \right) \begin{cases} U_{4n-1} V_{4n-2} U_3 - U_{12n-3} & \text{if } n \equiv 0 \pmod 3, \\ U_3 U_{4n-3} + U_{6n} V_{6n-3} & \text{if } n \equiv 1 \pmod 3, \\ -U_3 V_{2n} U_{6n-3} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

From (2.iii), we derive the generalized Fibonomial-Fibonacci-Lucas sum :

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ 3k+1 \end{Bmatrix}_U U_{3k} (-1)^{\binom{k}{2}} = (-1)^{n} \left(\prod_{t=1}^{2n} \frac{U_{3t}}{U_t} \right) \begin{cases} U_{2n} V_{2n+1} & \text{if } n \equiv 0 \pmod 3, \\ -U_{4n+1} & \text{if } n \equiv 1 \pmod 3, \\ 1 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

From (3.i), we derive the generalized Fibonomial-Fibonacci sum :

$$\sum_{k=0}^{4n} \begin{bmatrix} 4n \\ 3k+2 \end{bmatrix}_U U_{3k} (-1)^{\frac{1}{2}k(k-1)} = (-1)^{n} \left(\prod_{t=1}^{2n-1} \frac{U_{3t}}{U_t} \right) \begin{cases} V_1 & \text{if } n \equiv 0 \pmod 3, \\ U_{2n-2} V_{2n} & \text{if } n \equiv 1 \pmod 3, \\ -U_{4n-2} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

From (3.ii), we derive the generalized Fibonomial-Fibonacci sums corollary as a special case:

$$\sum_{k=0}^{4n+1} \begin{Bmatrix} 4n+1 \\ 3k+2 \end{Bmatrix}_U (-1)^{\binom{k}{2}} = (-1)^{n+1} \left(\prod_{t=1}^{2n} \frac{U_{3t}}{U_t} \right) \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3, \\ 1 & \text{if } n \equiv 1 \pmod 3, \\ -1 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

From (3.iii), we derive the generalized Fibonomial-Fibonacci-Lucas sum :

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ 3k+2 \end{Bmatrix}_U U_{9k} (-1)^{\binom{k}{2}} = (-1)^{n} \left(\prod_{t=1}^{2n-1} \frac{U_{3t}}{U_t} \right) \begin{cases} U_3 U_{8n-2} + U_{6n+3} V_{6n-3} & \text{if } n \equiv 0 \pmod 3, \\ U_{6n+3} V_{6n-3} - U_3 U_{4n-4} & \text{if } n \equiv 1 \pmod 3, \\ U_3 U_{6n-3} V_{2n+1} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

5. Conclusions

In this paper, we have considered triple aerated generalized Fibonomial sums with a general Fibonacci factor. There would not be any difficulty when one take a general Lucas number instead of the general Fibonacci number as a factor.

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