FORMULÆ RELATED TO THE q-DIXON FORMULA WITH APPLICATIONS TO FIBONOMIAL SUMS

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Abstract. The q-analogue of Dixon's identity involves three q-binomial coefficients as summands. We find many variations of it that have beautiful corollories in terms of Fibonomial sums. Proofs involve either several instances of the q-Dixon formula itself or are "mechanical," i. e., use the q-Zeilberger algorithm

1. Introduction

Define the second order linear sequence $\{U_n\}$ for $n \geq 2$ by

$$U_n = pU_{n-1} + U_{n-2}, \quad U_0 = 0,\ U_1 = 1.$$

For $n \geq k \geq 1$, define the generalized Fibonomial coefficient by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_U := \frac{U_1 U_2 \dots U_n}{(U_1 U_2 \dots U_k)(U_1 U_2 \dots U_{n-k})}$$

with $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\}_U = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\}_U = 1$. When $p = 1$, we obtain the usual Fibonomial coefficient, denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_F$. For more details about the Fibonomial and generalized Fibonomial coefficients, see [2, 3].

Our approach will be as follows. We will use the Binet forms

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{1 - q^n}{1 - q}$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$ where $\alpha, \beta = \left(p \pm \sqrt{p^2 + 4}\right)/2$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x; q)_n = (1 - x)(1 - xq) \dots (1 - xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

When $x = q$, we sometimes use the notation $(q)_n$ instead of $(q; q)_n$. We conveniently adopt the notation that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q = 0$ if $k < 0$ or $k > n$.

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_U = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \text{with} \quad q = -\alpha^{-2}.$$

We recall the q-analogue of Dixon's identity [1, 4], which is central in this paper:

$$\sum_k (-1)^k q^{\frac{k}{2}(3k+1)} \begin{bmatrix} a+b \\ a+k \end{bmatrix}_q \begin{bmatrix} b+c \\ b+k \end{bmatrix}_q \begin{bmatrix} c+a \\ c+k \end{bmatrix}_q = \frac{[a+b+c]!}{[a]![b]![c]!},$$

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where $[n]! = \prod_{i=1}^{n} \frac{1-q^i}{1-q} = (q;q)_n/(1-q)^n$.

Recently the authors of [5, 6] proved sum identities including certain generalized Fibonomial sums and their squares with or without the generalized Fibonacci and Lucas numbers. We recall such a result: if n and m are both *nonnegative* integers, then from [5], we have that

$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\}_U U_{(2m-1)k} = T_{n,m} \sum_{k=1}^{m} \left\{ {2m-1 \atop 2k-1} \right\}_U U_{(4k-2)n},$$

where

$$T_{n,m} = \begin{cases} \prod\limits_{k=0}^{n-m} V_{2k} & \text{if } \ n \geq m, \\ \prod\limits_{k=1}^{m-n-1} V_{2k}^{-1} & \text{if } \ n < m, \end{cases}$$

and three similar formulæ.

From [6], we have that for any positive integer n,

$$\sum_{k=0}^{2n} \mathbf{i}^{\pm k} \left\{ {2n \atop k} \right\}_U = \mathbf{i}^{\pm n} \prod_{k=1}^{n} V_{2k-1},$$

$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\}_U^2 = \prod_{k=1}^{n} \frac{V_{2k} U_{2(2k-1)}}{U_{2k}}$$

and

$$\sum_{k=0}^{n} (-1)^k \left\{ {2n+1 \atop 2k+1} \right\}_U = (-1)^{\binom{n}{2}} \begin{cases} \prod\limits_{k=1}^{n} V_k^2 & \text{if } n \text{ is odd}, \\ \prod\limits_{k=1}^{n} V_{2k} & \text{if } n \text{ is even}. \end{cases}$$

In this paper, we consider some sum formulæ whose terms include certain triple Fibonomial coefficients, with or without extra Fibonacci numbers. To be systematic, we first organize the q-Dixon type identities in a list, then discuss the proofs of them, and then get a list of Fibonacci type identities as corollaries.

2. Triple Gaussian q-Binomial Sums

The identities in this section hold for all nonnegative integers n.

(1)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-1)} = (-1)^n q^{-\frac{n}{2}(3n+1)} \begin{bmatrix} 2n \\ n \end{bmatrix}_q \begin{bmatrix} 3n+1 \\ n \end{bmatrix}_q .$$

(2)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} (1+q^{2k}) = 2(-1)^n q^{-\frac{n}{2}(3n+1)} \begin{bmatrix} 2n \\ n \end{bmatrix}_q \begin{bmatrix} 3n+1 \\ n \end{bmatrix}_q .$$

(3)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} (1-q^{2k}) = 2(-1)^n q^{-\frac{n}{2}(3n+1)} (1-q^{2n+1}) \begin{bmatrix} 2n \\ n \end{bmatrix}_q \begin{bmatrix} 3n \\ n-1 \end{bmatrix}_q .$$

(4)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+2 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} \left(1-q^k\right) = (-1)^n (1-q)\, q^{-\frac{n}{2}(3n+1)} \begin{bmatrix} 2n \\ n+1 \end{bmatrix}_q \begin{bmatrix} 3n+1 \\ n \end{bmatrix}_q .$$

(5)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+3 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-5)} \left(1-q^k\right)^2 = (-1)^{n+1} q^{-\frac{1}{2}(n+1)(3n+2)} \frac{\left(1-q^{2n}\right)\left(1-q^{2n+3}\right)\left(1-q^{2n+4}\right)}{(1-q^{n-1})} \begin{bmatrix} 2n \\ n-2 \end{bmatrix}_q \begin{bmatrix} 3n+1 \\ n-1 \end{bmatrix}_q .$$

(6)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+3 \\ k+1 \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-1)} = (-1)^n q^{-\frac{n}{2}(3n+1)} \frac{1-q^{2n+3}}{1-q^n} \begin{bmatrix} 2n \\ n-1 \end{bmatrix}_q \begin{bmatrix} 3n+2 \\ n \end{bmatrix}_q .$$

(7)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+3 \\ k+1 \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-5)} \left(1-q^{2k}\right)^2 = (-1)^{n+1} q^{-\frac{1}{2}(n+1)(3n+2)} \left(1-q^{2n}\right)\left(1-q^{2n+1}\right)\left(1-q^{2n+2}\right)\left(1-q^{2n+3}\right) \times \begin{bmatrix} 2n \\ n-1 \end{bmatrix}_q \begin{bmatrix} 3n \\ n-1 \end{bmatrix}_q .$$

(8)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+3 \\ k+1 \end{bmatrix}_q (-1)^k q^{\frac{3}{2}k(k-2n-1)} \left(1-q^{2k}\right) = 2(-1)^n q^{-\frac{n}{2}(3n+1)} \left(1-q^{2n+3}\right) \begin{bmatrix} 2n \\ n-1 \end{bmatrix}_q \begin{bmatrix} 3n+1 \\ n \end{bmatrix}_q .$$

(9)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+3 \\ k+1 \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} \left(1+q^k\right)^2 = 4(-1)^n q^{-\frac{n}{2}(3n+1)} \frac{1-q^{2n+3}}{1-q^n} \begin{bmatrix} 2n \\ n-1 \end{bmatrix}_q \begin{bmatrix} 3n+2 \\ n \end{bmatrix}_q .$$

(10)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+3 \\ k+2 \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n+1)} = (-1)^n q^{-\frac{n}{2}(3n-1)} \frac{1-q^{2n+3}}{1-q^n} \begin{bmatrix} 2n \\ n-1 \end{bmatrix}_q \begin{bmatrix} 3n+2 \\ n \end{bmatrix}_q .$$

(11)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+3 \\ k+2 \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-1)} \left(1-q^k\right)^2 = (-1)^n q^{-\frac{n}{2}(3n+1)} \frac{\left(1-q^{2n}\right)\left(1-q^{2n+3}\right)\left(1-q^{2n+4}\right)}{\left(1-q^{n-1}\right)} \begin{bmatrix} 2n \\ n-2 \end{bmatrix}_q \begin{bmatrix} 3n+1 \\ n-1 \end{bmatrix}_q .$$

(12)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+4 \\ k+1 \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} \left(1-q^k\right) = (-1)^n q^{-\frac{n}{2}(3n+1)} \left(1-q^2\right) \frac{1-q^{2n+4}}{1-q^{n-1}} \begin{bmatrix} 2n \\ n-2 \end{bmatrix}_q \begin{bmatrix} 3n+2 \\ n \end{bmatrix}_q .$$

(13)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+2 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-5)} \left(1-q^k\right) = (-1)^{n+1} q^{-\frac{1}{2}(n+1)(3n+2)} \left(1-q^{2n+2}\right) \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q \begin{bmatrix} 3n+2 \\ n \end{bmatrix}_q .$$

(14)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n+2 \\ k \end{bmatrix}_q^2 (-1)^k q^{\frac{k}{2}(3k-6n-5)} = (-1)^{n+1} q^{-\frac{1}{2}(n+1)(3n+2)} \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q \begin{bmatrix} 3n+3 \\ n+1 \end{bmatrix}_q .$$

(15)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n+2 \\ k \end{bmatrix}_q^2 (-1)^k q^{\frac{k}{2}(3k-6n-7)} \left(1-q^k\right)^2$$
$$= (-1)^{n+1} q^{-\frac{1}{2}(n+1)(3n+4)} \left(1-q^{2n+2}\right)^2 \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q \begin{bmatrix} 3n+2 \\ n \end{bmatrix}_q .$$

(16)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+2 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-5)} \left(1+q^k\right)$$
$$= (-1)^{n+1} q^{-\frac{1}{2}(n+1)(3n+2)} \left(1+q^{2n+2}\right) \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q \begin{bmatrix} 3n+2 \\ n \end{bmatrix}_q .$$

(17)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+3 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-5)}$$
$$= (-1)^{n+1} q^{-\frac{1}{2}(n+1)(3n+2)} \left(1+q^{n+2}\right) \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q \begin{bmatrix} 3n+3 \\ n \end{bmatrix}_q .$$

(18)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+3 \\ k+1 \end{bmatrix}_q (-1)^k q^{-\frac{1}{2}(3k+1)(2-k+2n)} \left(1-q^{2k}\right)$$
$$= (-1)^{n+1} q^{-\frac{1}{2}(n+1)(3n+4)} \left(1+q^{2n+1}\right) \left(1-q^{2n+3}\right) \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q \begin{bmatrix} 3n+2 \\ n \end{bmatrix}_q .$$

(19)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+4 \\ k+2 \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} \left(1-q^k\right)$$
$$= (-1)^{n+1} \left(1-q^2\right) q^{-\frac{n}{2}(3n+1)} \frac{1-q^{2n+4}}{1-q^n} \begin{bmatrix} 2n+1 \\ n-1 \end{bmatrix}_q \begin{bmatrix} 3n+3 \\ n \end{bmatrix}_q .$$

(20)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+4 \\ k+2 \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} \left(1+q^k\right)$$
$$= (-1)^{n} \left(1+q^2\right) q^{-\frac{n}{2}(3n+1)} \frac{1-q^{2n+4}}{1-q^n} \begin{bmatrix} 2n+1 \\ n-1 \end{bmatrix}_q \begin{bmatrix} 3n+3 \\ n \end{bmatrix}_q .$$

(21)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+4 \\ k+1 \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-7)} \left(1-q^k\right)^3$$

$$= (-1)^{n+1} q^{-\frac{1}{2}(n+1)(3n+4)} \frac{\left(1-q^{2n+1}\right)\left(1-q^{2n+3}\right)\left(1-q^{2n+4}\right)\left(1-q^{n+1}\right)}{(1-q^n)}$$

$$\times \begin{bmatrix} 2n+1 \\ n-1 \end{bmatrix}_q \begin{bmatrix} 3n+2 \\ n \end{bmatrix}_q.$$

(22)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n+3 \\ k \end{bmatrix}_q^2 (-1)^k q^{\frac{k}{2}(3k-6n-9)} \left(1-q^k\right)^2$$

$$= (-1)^{n+1} (1-q) q^{-\frac{1}{2}\left(3(n+1)^2+n+1+2\right)} \begin{bmatrix} 2n+3 \\ n+1 \end{bmatrix}_q \begin{bmatrix} 3n+3 \\ n+1 \end{bmatrix}_q \frac{1-q^{n+2}}{1+q^{n+1}}.$$

(23)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n+3 \\ k \end{bmatrix}_q^2 (-1)^k q^{\frac{k}{2}(3k-6n-7)}$$

$$= (-1)^{n+1} q^{-\frac{1}{2}(n+1)(3n+4)} \begin{bmatrix} 2n+2 \\ n+1 \end{bmatrix}_q \begin{bmatrix} 3n+4 \\ n+1 \end{bmatrix}_q.$$

(24)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+5 \\ k+1 \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-5)}$$

$$= (-1)^{n+1} q^{-\frac{1}{2}(n+1)(3n+2)} \frac{\left(1-q^{2n+5}\right)\left(1-q^{2n+6}\right)}{\left(1-q^{n-1}\right)(1-q^n)} \begin{bmatrix} 2n+1 \\ n-2 \end{bmatrix}_q \begin{bmatrix} 3n+4 \\ n \end{bmatrix}_q.$$

(25)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+5 \\ k+1 \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-7)} \left(1-q^k\right)^2 = (-1)^n$$

$$\times q^{-\frac{1}{2}(n+1)(3n+4)} \frac{\left(1-q^{2n+1}\right)\left(1-q^{2n+4}\right)\left(1-q^{2n+5}\right)}{(1-q^n)} \begin{bmatrix} 2n+1 \\ n-1 \end{bmatrix}_q \begin{bmatrix} 3n+3 \\ n \end{bmatrix}_q.$$

3. Proofs

In this section we choose some of the identities given in the previous Section and prove them. We prove the identities 1, 14, 13, 15, 3 and 2, respectively.

Proof of identity 1.

First if we replace $k \to n-k$, then we write

$$\sum_k \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ n-k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k+1)} = \begin{bmatrix} 2n \\ n \end{bmatrix}_q \begin{bmatrix} 3n+1 \\ n \end{bmatrix}_q,$$

which is an equivalent form of identity (1). Another equivalent form is

$$\sum_k (1-q^{2n+1}) \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k+1)} = \frac{(q)_{3n+1}}{(q)_n^3},$$

and this one we will prove now by two applications of Dixon's formula. Note that within the following computations, we sometimes change $k \leftrightarrow -k$ in order to transform the exponent $\frac{k(3k-1)}{2}$ to $\frac{k(3k+1)}{2}$.

$$\begin{aligned}
&\sum_k (1-q^{2n+1}) \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k+1)} \\
&= \sum_k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q (1-q^{n+1-k})(-1)^k q^{\frac{k}{2}(3k+1)} \\
&= \frac{(q)_{3n+1}}{(q)_n^2 q_{n+1}} - \sum_k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q q^{n+1-k}(-1)^k q^{\frac{k}{2}(3k+1)} \\
&= \frac{(q)_{3n+1}}{(q)_n^2 (q)_{n+1}} - q^{n+1} \sum_k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k+1)} \\
&= \frac{(q)_{3n+1}}{(q)_n^2 (q)_{n+1}} - q^{n+1} \frac{(q)_{3n+1}}{(q)_n^2 (q)_{n+1}} \\
&= \frac{(q)_{3n+1}}{(q)_n^3}.
\end{aligned}$$

Proof of identity 14. By taking $k \to n+1-k$ and after some rearrangements, then we write

$$\sum_k (1-q^{2n+2}) \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_q^2 (-1)^k q^{\frac{k}{2}(3k-1)} = \frac{(q)_{3n+3}}{(q)_n (q)_{n+1}^2}.$$

This form is equivalent to identity (5) and will be proved now by two applications of Dixon's identity.

$$\begin{aligned}
&\sum_k (1-q^{2n+2}) \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_q^2 (-1)^k q^{\frac{k}{2}(3k-1)} \\
&= \sum_k (1-q^{n+1+k}) \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_q^3 (-1)^k q^{\frac{k}{2}(3k-1)} \\
&= \frac{(q)_{3n+3}}{(q)_{n+1}^3} - q^{n+1} \sum_k \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_q^3 (-1)^k q^{\frac{k}{2}(3k+1)} \\
&= \frac{(q)_{3n+3}}{(q)_{n+1}^3} - q^{n+1} \frac{(q)_{3n+3}}{(q)_{n+1}^3} \\
&= \frac{(q)_{3n+3}}{(q)_n (q)_{n+1}^2}.
\end{aligned}$$

Proof of identity 13. By replacing $k\to n+1+k$ and rearrangements, we get the equivalent form

$$\sum_k \begin{bmatrix}2n+2\\n+1+k\end{bmatrix}_q \begin{bmatrix}2n+1\\n+1+k\end{bmatrix}_q \begin{bmatrix}2n+1\\n+k\end{bmatrix}_q \times(-1)^k q^{\frac{k}{2}(3k+1)}(1-q^{n+1-k}) = \frac{(q)_{3n+2}}{(q)_n^2(q)_{n+1}}.$$

It will be proved by two applications of Dixon's formula:

$$\begin{aligned}
&\sum_k \begin{bmatrix}2n+2\\n+1+k\end{bmatrix}_q \begin{bmatrix}2n+1\\n+1+k\end{bmatrix}_q \begin{bmatrix}2n+1\\n+k\end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k+1)}(1-q^{n+1-k})\\
&= \frac{(q)_{3n+2}}{(q)_n(q)_{n+1}^2} - q^{n+1}\sum_k \begin{bmatrix}2n+2\\n+1+k\end{bmatrix}_q \begin{bmatrix}2n+1\\n+1+k\end{bmatrix}_q \begin{bmatrix}2n+1\\n+k\end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-1)}\\
&= \frac{(q)_{3n+2}}{(q)_n(q)_{n+1}^2} - q^{n+1}\frac{(q)_{3n+2}}{(q)_n(q)_{n+1}^2}\\
&= \frac{(q)_{3n+2}}{(q)_n^2(q)_{n+1}}.
\end{aligned}$$

Proof of identity 15.

By taking $k\to n+1-k$ and some rearrangements, the claimed identity takes the equivalent form

$$\sum_k (1-q^{2n+2}) \begin{bmatrix}2n+1\\n+k\end{bmatrix}_q \begin{bmatrix}2n+1\\n+1+k\end{bmatrix}_q^2 (-1)^k q^{\frac{k}{2}(3k+1)} = \frac{(q)_{3n+2}}{(q)_n^2(q)_{n+1}},$$

which will be proved by Dixon's formula:

$$\begin{aligned}
&\sum_k (1-q^{2n+2}) \begin{bmatrix}2n+1\\n+k\end{bmatrix}_q \begin{bmatrix}2n+1\\n+1+k\end{bmatrix}_q^2 (-1)^k q^{\frac{k}{2}(3k+1)}\\
&= \sum_k \begin{bmatrix}2n+1\\n+k\end{bmatrix}_q \begin{bmatrix}2n+2\\n+1+k\end{bmatrix}_q \begin{bmatrix}2n+1\\n+1+k\end{bmatrix}_q (-1)^k (1-q^{n+1-k}) q^{\frac{k}{2}(3k+1)}\\
&= \frac{(q)_{3n+2}}{(q)_n(q)_{n+1}^2} - q^{n+1}\sum_k \begin{bmatrix}2n+1\\n+k\end{bmatrix}_q \begin{bmatrix}2n+2\\n+1+k\end{bmatrix}_q \begin{bmatrix}2n+1\\n+1+k\end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-1)}\\
&= \frac{(q)_{3n+2}}{(q)_n(q)_{n+1}^2} - q^{n+1}\sum_k \begin{bmatrix}2n+1\\n+k+1\end{bmatrix}_q \begin{bmatrix}2n+2\\n+1+k\end{bmatrix}_q \begin{bmatrix}2n+1\\n+k\end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k+1)}\\
&= \frac{(q)_{3n+2}}{(q)_n(q)_{n+1}^2} - q^{n+1}\frac{(q)_{3n+2}}{(q)_n(q)_{n+1}^2}\\
&= \frac{(q)_{3n+2}}{(q)_n^2(q)_{n+1}}.
\end{aligned}$$

Proof of identity 3. This proof is more involved and requires auxiliary quantities that will be evaluated by several applications of Dixon's identity. Define

$$T := \sum_k \begin{bmatrix}2n\\k\end{bmatrix}_q^2 \begin{bmatrix}2n+1\\k\end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} q^k,$$

$$W := \sum_k \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} q^{2k}$$

and

$$X := \sum_k \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)}.$$

To complete the proof we should prove that

$$X - W = 2(-1)^n q^{-\frac{n}{2}(3n+1)} (1-q^{2n+1}) \begin{bmatrix} 2n \\ n \end{bmatrix}_q \begin{bmatrix} 3n \\ n-1 \end{bmatrix}_q.$$

First we notice that T is the sum in identity (1), so

$$T = (-1)^n q^{-\frac{n}{2}(3n+1)} \frac{1}{1-q^{2n+1}} \frac{(q)_{3n+1}}{(q)_n (q)_n (q)_n}.$$

Next we compute

$$\begin{aligned}
V &= \sum_k \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} (1-q^k)^2 \\
&= (1-q^{2n})(1-q^{2n+1}) \sum_k (-1)^k q^{\frac{k}{2}(3k-6n-3)} \begin{bmatrix} 2n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n \\ k-1 \end{bmatrix}_q \\
&= (1-q^{2n})(1-q^{2n+1}) \sum_k (-1)^k q^{\frac{k}{2}(3k-6n-3)} \begin{bmatrix} 2n-1 \\ 2n-k \end{bmatrix}_q \begin{bmatrix} 2n \\ 2n-k \end{bmatrix}_q \begin{bmatrix} 2n \\ 2n+1-k \end{bmatrix}_q \\
&= (1-q^{2n})(1-q^{2n+1}) \sum_j (-1)^{j-1} q^{\frac{j}{2}(3j-6n-3)} \begin{bmatrix} 2n-1 \\ j-1 \end{bmatrix}_q \begin{bmatrix} 2n \\ j \end{bmatrix}_q \begin{bmatrix} 2n \\ j-1 \end{bmatrix}_q \\
&= -V,
\end{aligned}$$

hence $V = 0$. Therefore we get

$$\begin{aligned}
&\sum_k \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} (1-q^k) \\
&\quad = \sum_k \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} (1-q^k) q^k
\end{aligned}$$

and thus

$$X - T = T - W$$

and so

$$X + W = 2T,$$

which will be used later. Now we compute

$$\begin{aligned}
W &= \sum_k (-1)^k q^{\frac{k}{2}(3k-6n-3)} q^{2k} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \\
&= (-1)^n q^{-\frac{n}{2}(3n-1)} \sum_k (-1)^k q^{\frac{k}{2}(3k+1)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q
\end{aligned}$$

$$= (-1)^n q^{-\frac{n}{2}(3n-1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k+1)} (1-q^{n+1+k}) \times \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$

$$= (-1)^n q^{-\frac{n}{2}(3n-1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k+1)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$

$$- (-1)^n q^{-\frac{n}{2}(3n-1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k+1)} q^{n+1+k} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$

$$= (-1)^n q^{-\frac{n}{2}(3n-1)} \frac{1}{1-q^{2n+1}} \frac{(q)_{3n+1}}{(q)_n (q)_n (q)_{n+1}}$$

$$- (-1)^n q^{-\frac{n}{2}(3n-1)+\frac{n}{2}+1} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k+3)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$

and

$$X = \sum_k (-1)^k q^{\frac{k}{2}(3k-6n-3)} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q$$

$$= (-1)^n q^{-\frac{3}{2}n(n+1)} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$

$$= (-1)^n \frac{q^{-\frac{3}{2}n(n+1)}}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} (1-q^{n+k+1}) \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$

$$= (-1)^n q^{-\frac{3}{2}n(n+1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$

$$- (-1)^n q^{-\frac{3}{2}n(n+1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} q^{n+k+1} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$

$$= (-1)^n q^{-\frac{3}{2}n(n+1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$

$$- (-1)^n q^{-\frac{1}{2}(3n-2)(n+1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-1)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q,$$

which by $k \to -k$ in the second sum, equals

$$= (-1)^n q^{-\frac{3}{2}n(n+1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$

$$- (-1)^n q^{-\frac{1}{2}(3n-2)(n+1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k+1)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q$$

$$= (-1)^n q^{-\frac{3}{2}n(n+1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$

$$- (-1)^n q^{-\frac{1}{2}(3n-2)(n+1)} \frac{1}{1-q^{2n+1}} \frac{(q)_{3n+1}}{(q)_n (q)_n (q)_{n+1}}.$$

Consequently we have the summarized results

$$\begin{aligned}W &= (-1)^n q^{-\frac{n}{2}(3n-1)} \frac{1}{1-q^{2n+1}} \frac{(q)_{3n+1}}{(q)_n (q)_n (q)_{n+1}} \\ &\quad - (-1)^n q^{-\frac{n}{2}(3n-1)+n+1} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q\end{aligned}$$

and

$$\begin{aligned}X &= (-1)^n q^{-\frac{3}{2}n(n+1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q \\ &\quad - (-1)^n q^{-\frac{1}{2}(3n-1)(n+1)} \frac{1}{1-q^{2n+1}} \frac{(q)_{3n+1}}{(q)_n (q)_n (q)_{n+1}}.\end{aligned}$$

Therefore

$$\begin{aligned}&q^{3n+1} X + W \\ &= -(-1)^n q^{-\frac{1}{2}(3n-2)(n+1)} q^{3n+1} \frac{1}{1-q^{2n+1}} \frac{(q)_{3n+1}}{(q)_n (q)_n (q)_{n+1}} \\ &\qquad + (-1)^n q^{-\frac{n}{2}(3n-1)} \frac{1}{1-q^{2n+1}} \frac{(q)_{3n+1}}{(q)_n (q)_n (q)_{n+1}} \\ &= (-1)^n q^{-\frac{n}{2}(3n-1)} \frac{1-q^{2n+2}}{1-q^{2n+1}} \frac{(q)_{3n+1}}{(q)_n (q)_n (q)_{n+1}}.\end{aligned}$$

We can rewrite this as

$$q^{3n+1} X + W = T(1+q^{n+1}) q^n.$$

But we also know that

$$W + X = 2T.$$

From these two relations, we can compute X and W and thus $X - W$ as

$$\begin{aligned}X &= T \frac{1}{1-q^{3n+1}} \Big(2 - (1+q^{n+1}) q^n \Big) \\ &= (-1)^n q^{-\frac{n}{2}(3n+1)} \Big(2 - (1+q^{n+1}) q^n \Big) \frac{1}{1-q^{2n+1}} \frac{(q)_{3n}}{(q)_n (q)_n (q)_n}\end{aligned}$$

and

$$\begin{aligned}W &= 2T - X \\ &= (-1)^n q^{-\frac{n}{2}(3n+1)+n} (1 + q^{n+1} - 2q^{2n+1}) \frac{1}{1-q^{3n+1}} \frac{1}{1-q^{2n+1}} \frac{(q)_{3n+1}}{(q)_n (q)_n (q)_n},\end{aligned}$$

and so the result

$$\begin{aligned}X - W &= T \frac{1}{1-q^{3n+1}} \Big(2 - (1+q^{n+1}) q^n \Big) - T \frac{1}{1-q^{3n+1}} q^n (1 + q^{n+1} - 2q^{2n+1}) \\ &= T \frac{1}{1-q^{3n+1}} \Big((2 - (1+q^{n+1}) q^n) - q^n (1 + q^{n+1} - 2q^{2n+1}) \Big) \\ &= 2T (1-q^n)(1-q^{2n+1}) \frac{1}{1-q^{3n+1}} \\ &= 2(-1)^n q^{-\frac{n}{2}(3n+1)} (1-q^n)(1-q^{2n+1}) \frac{1}{1-q^{2n+1}} \frac{(q)_{3n}}{(q)_n (q)_n (q)_n}\end{aligned}$$

$$= 2(-1)^n q^{-\frac{n}{2}(3n+1)} \frac{(q)_{3n}}{(q)_n (q)_n (q)_{n-1}},$$

as claimed.

Remark. From this proof we know that

$$X + W = 2T,$$

which proves the identity 2.

As the last example has shown, the reduction to instances of the q-Dixon identity can be quite involved. Therefore we present an alternative method, namely the q-Zeilberger algorithm [7]. We discuss identity 6 as a showcase: Define

$$T_n := \sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+3 \\ k+1 \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-1)}.$$

Zeilberger's algorithm produces a recursion

$$a_n T_n + b_n T_{n+1} + c_n T_{n+2} + d_n T_{n+3} = 0,$$

where a_n, b_n, c_n, d_n are complicated expressions with about 1000 terms each.

Set

$$U_n := (-1)^n q^{-\frac{n}{2}(3n+1)} \frac{1-q^{2n+3}}{1-q^n} \begin{bmatrix} 2n \\ n-1 \end{bmatrix}_q \begin{bmatrix} 3n+2 \\ n \end{bmatrix}_q,$$

then it can be checked (by a computer) that also

$$a_n U_n + b_n U_{n+1} + c_n U_{n+2} + d_n U_{n+3} = 0.$$

After checking a few initial values directly, this proves indeed that $T_n = U_n$ for all nonnegative integers n.

4. Applications to the Fibonomials Sums Identities

In this section, we present corollaries of our previous list of identities, by specializing the value of q as described in the Introduction. Each identity corresponds now to two identities which have slightly different forms. By replacing $n \to 2n$, we get a formula labelled with "e" (even), and by replacing $n \to 2n+1$, we get a formula labelled with "o" (odd).

1-e)

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+3 \\ k \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k+1)} = (-1)^{n+1} \begin{Bmatrix} 4n+2 \\ 2n+1 \end{Bmatrix}_U \begin{Bmatrix} 6n+4 \\ 2n+1 \end{Bmatrix}_U.$$

1-o)

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+1 \\ k \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k-1)} = (-1)^n \begin{Bmatrix} 4n \\ 2n \end{Bmatrix}_U \begin{Bmatrix} 6n+1 \\ 2n \end{Bmatrix}_U,$$

2-e)

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+1 \\ k \end{Bmatrix}_U V_{2k} (-1)^{\frac{1}{2}k(k+1)} = 2(-1)^n \begin{Bmatrix} 4n \\ 2n \end{Bmatrix}_U \begin{Bmatrix} 6n+1 \\ 2n \end{Bmatrix}_U,$$

2-o)

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+3 \\ k \end{Bmatrix}_U V_{2k}(-1)^{\frac{1}{2}k(k-1)} = 2(-1)^{n+1} \begin{Bmatrix} 4n+2 \\ 2n+1 \end{Bmatrix}_U \begin{Bmatrix} 6n+4 \\ 2n+1 \end{Bmatrix}_U.$$

3-e)

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+1 \\ k \end{Bmatrix}_U U_{2k}(-1)^{\frac{k}{2}(k+1)} = 2(-1)^n U_{4n+1} \begin{Bmatrix} 4n \\ 2n \end{Bmatrix}_U \begin{Bmatrix} 6n \\ 2n-1 \end{Bmatrix}_U,$$

3-o)

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+3 \\ k \end{Bmatrix}_U U_{2k}(-1)^{\frac{1}{2}k(k-1)}$$
$$= 2(-1)^{n+1} U_{4n+3} \begin{Bmatrix} 4n+2 \\ 2n+1 \end{Bmatrix}_U \begin{Bmatrix} 6n+3 \\ 2n \end{Bmatrix}_U.$$

4-e)

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U U_k(-1)^{\frac{1}{2}k(k+1)} = (-1)^n \begin{Bmatrix} 4n \\ 2n+1 \end{Bmatrix}_U \begin{Bmatrix} 6n+1 \\ 2n \end{Bmatrix}_U,$$

4-o)

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+4 \\ k \end{Bmatrix}_U U_k\,(-1)^{\frac{1}{2}k(k-1)} = (-1)^{n+1} \begin{Bmatrix} 4n+2 \\ 2n+2 \end{Bmatrix}_U \begin{Bmatrix} 6n+4 \\ 2n+1 \end{Bmatrix}_U.$$

5-e)

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+3 \\ k \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k-1)} U_k^2$$
$$= (-1)^n \frac{U_{4n}U_{4n+3}U_{4n+4}}{U_{2n-1}} \begin{Bmatrix} 4n \\ 2n-2 \end{Bmatrix}_U \begin{Bmatrix} 6n+1 \\ 2n-1 \end{Bmatrix}_U,$$

5-o)

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+5 \\ k \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k+1)} U_k^2$$
$$= (-1)^{n+1} \frac{U_{4n+2}U_{4n+5}U_{4n+6}}{U_{2n}} \begin{Bmatrix} 4n+2 \\ 2n-1 \end{Bmatrix}_U \begin{Bmatrix} 6n+4 \\ 2n \end{Bmatrix}_U.$$

6-e)

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+3 \\ k+1 \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k-1)} = (-1)^n \frac{U_{4n+3}}{U_{2n}} \begin{Bmatrix} 4n \\ 2n-1 \end{Bmatrix}_U \begin{Bmatrix} 6n+2 \\ 2n \end{Bmatrix}_U,$$

6-o)

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+5 \\ k+1 \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k+1)} = (-1)^{n+1} \frac{U_{4n+5}}{U_{2n+1}} \begin{Bmatrix} 4n+2 \\ 2n \end{Bmatrix}_U \begin{Bmatrix} 6n+5 \\ 2n+1 \end{Bmatrix}_U.$$

7-e)

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+3 \\ k+1 \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k-1)} U_{2k}^2$$
$$= (-1)^n \Delta U_{4n} U_{4n+1} U_{4n+2} U_{4n+3} \begin{Bmatrix} 4n \\ 2n-1 \end{Bmatrix}_U \begin{Bmatrix} 6n \\ 2n-1 \end{Bmatrix}_U,$$

7-o)

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+5 \\ k+1 \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k+1)} U_{2k}^2$$
$$= (-1)^{n+1} \Delta U_{4n+2} U_{4n+3} U_{4n+4} U_{4n+5} \begin{Bmatrix} 4n+2 \\ 2n \end{Bmatrix}_U \begin{Bmatrix} 6n+3 \\ 2n \end{Bmatrix}_U.$$

8-e)

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+3 \\ k+1 \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k+1)} U_{2k}$$
$$= 2(-1)^n U_{4n+3} \begin{Bmatrix} 4n \\ 2n-1 \end{Bmatrix}_U \begin{Bmatrix} 6n+1 \\ 2n \end{Bmatrix}_U,$$

8-o)

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+5 \\ k+1 \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k-1)} U_{2k}$$
$$= 2(-1)^{n+1} U_{4n+5} \begin{Bmatrix} 4n+2 \\ 2n \end{Bmatrix}_U \begin{Bmatrix} 6n+4 \\ 2n+1 \end{Bmatrix}_U.$$

9-e)

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+3 \\ k+1 \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k+1)} V_{2k}^2$$
$$= 4(-1)^n \frac{U_{4n+3}}{U_{2n}} \begin{Bmatrix} 4n \\ 2n-1 \end{Bmatrix}_U \begin{Bmatrix} 6n+2 \\ 2n \end{Bmatrix}_U,$$

9-o)

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+5 \\ k+1 \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k-1)} V_{2k}^2$$
$$= 4(-1)^{n+1} \frac{U_{4n+5}}{U_{2n+1}} \begin{Bmatrix} 4n+2 \\ 2n \end{Bmatrix}_U \begin{Bmatrix} 6n+5 \\ 2n+1 \end{Bmatrix}_U.$$

10-e)

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+3 \\ k+2 \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k+1)}$$
$$= (-1)^n \frac{U_{4n+3}}{U_{2n}} \begin{Bmatrix} 4n \\ 2n-1 \end{Bmatrix}_U \begin{Bmatrix} 6n+2 \\ 2n \end{Bmatrix}_U,$$

10-o)

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+5 \\ k+2 \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k-1)} = (-1)^{n+1} \frac{U_{4n+5}}{U_{2n+1}} \begin{Bmatrix} 4n+2 \\ 2n \end{Bmatrix}_U \begin{Bmatrix} 6n+5 \\ 2n+1 \end{Bmatrix}_U .$$

11-e)

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+3 \\ k+2 \end{Bmatrix}_U U_k^2 (-1)^{\frac{1}{2}k(k-1)} = (-1)^n \frac{U_{4n} U_{4n+3} U_{4n+4}}{U_{2n-1}} \begin{Bmatrix} 4n \\ 2n-2 \end{Bmatrix}_U \begin{Bmatrix} 6n+1 \\ 2n-1 \end{Bmatrix}_U ,$$

11-o)

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+5 \\ k+2 \end{Bmatrix}_U U_k^2 (-1)^{\frac{1}{2}k(k+1)} = (-1)^{n+1} \frac{U_{4n+2} U_{4n+5} U_{4n+6}}{U_{2n}} \begin{Bmatrix} 4n+2 \\ 2n-1 \end{Bmatrix}_U \begin{Bmatrix} 6n+4 \\ 2n \end{Bmatrix}_U .$$

12-e)

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+4 \\ k+1 \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k+1)} U_k = (-1)^n V_1 \frac{U_{4n+4}}{U_{2n-1}} \begin{Bmatrix} 4n \\ 2n-2 \end{Bmatrix}_U \begin{Bmatrix} 6n+2 \\ 2n \end{Bmatrix}_U ,$$

12-o)

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+6 \\ k+1 \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k-1)} U_k = (-1)^{n+1} V_1 \frac{U_{4n+6}}{U_{2n}} \begin{Bmatrix} 4n+2 \\ 2n-1 \end{Bmatrix}_U \begin{Bmatrix} 6n+5 \\ 2n+1 \end{Bmatrix}_U .$$

13-e)

$$\sum_{k=0}^{4n+1} \begin{Bmatrix} 4n+1 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U U_k (-1)^{\frac{1}{2}k(k-1)} = (-1)^n U_{4n+2} \begin{Bmatrix} 4n+1 \\ 2n \end{Bmatrix}_U \begin{Bmatrix} 6n+2 \\ 2n \end{Bmatrix}_U ,$$

13-o)

$$\sum_{k=0}^{4n+3} \begin{Bmatrix} 4n+3 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+4 \\ k \end{Bmatrix}_U U_k (-1)^{\frac{1}{2}k(k+1)} = (-1)^{n+1} U_{4n+4} \begin{Bmatrix} 4n+3 \\ 2n+1 \end{Bmatrix}_U \begin{Bmatrix} 6n+5 \\ 2n+1 \end{Bmatrix}_U .$$

14-e)

$$\sum_{k=0}^{4n+1}\begin{Bmatrix}4n+1\\k\end{Bmatrix}_U\begin{Bmatrix}4n+2\\k\end{Bmatrix}_U^2(-1)^{\frac{1}{2}k(k-1)}=(-1)^n\begin{Bmatrix}4n+1\\2n\end{Bmatrix}_U\begin{Bmatrix}6n+3\\2n+1\end{Bmatrix}_U,$$

14-o)

$$\sum_{k=0}^{4n+3}\begin{Bmatrix}4n+3\\k\end{Bmatrix}_U\begin{Bmatrix}4n+4\\k\end{Bmatrix}_U^2(-1)^{\frac{1}{2}k(k+1)}=(-1)^{n+1}\begin{Bmatrix}4n+3\\2n+1\end{Bmatrix}_U\begin{Bmatrix}6n+6\\2n+2\end{Bmatrix}_U.$$

15-e)

$$\sum_{k=0}^{4n+1}\begin{Bmatrix}4n+1\\k\end{Bmatrix}_U\begin{Bmatrix}4n+2\\k\end{Bmatrix}_U^2U_k^2\,(-1)^{\frac{1}{2}k(k+1)}$$
$$=(-1)^nU_{4n+2}^2\begin{Bmatrix}4n+1\\2n\end{Bmatrix}_U\begin{Bmatrix}6n+2\\2n\end{Bmatrix}_U,$$

15-o)

$$\sum_{k=0}^{4n+3}\begin{Bmatrix}4n+3\\k\end{Bmatrix}_U\begin{Bmatrix}4n+4\\k\end{Bmatrix}_U^2U_k^2\,(-1)^{\frac{1}{2}k(k-1)}$$
$$=(-1)^{n+1}U_{4n+4}^2\begin{Bmatrix}4n+3\\2n+1\end{Bmatrix}_U\begin{Bmatrix}6n+5\\2n+1\end{Bmatrix}_U.$$

16-e)

$$\sum_{k=0}^{4n+1}\begin{Bmatrix}4n+1\\k\end{Bmatrix}_U^2\begin{Bmatrix}4n+2\\k\end{Bmatrix}_UV_k\,(-1)^{\frac{1}{2}k(k+1)}$$
$$=(-1)^nV_{4n+2}\begin{Bmatrix}4n+1\\2n\end{Bmatrix}_U\begin{Bmatrix}6n+2\\2n\end{Bmatrix}_U,$$

16-o)

$$\sum_{k=0}^{4n+3}\begin{Bmatrix}4n+3\\k\end{Bmatrix}_U^2\begin{Bmatrix}4n+4\\k\end{Bmatrix}_UV_k\,(-1)^{\frac{1}{2}k(k-1)}$$
$$=(-1)^{n+1}V_{4n+4}\begin{Bmatrix}4n+3\\2n+1\end{Bmatrix}_U\begin{Bmatrix}6n+5\\2n+1\end{Bmatrix}_U.$$

17-e)

$$\sum_{k=0}^{4n+1}\begin{Bmatrix}4n+1\\k\end{Bmatrix}_U^2\begin{Bmatrix}4n+3\\k\end{Bmatrix}_U(-1)^{\frac{1}{2}k(k+1)}$$
$$=(-1)^nV_{2n+2}\begin{Bmatrix}4n+1\\2n\end{Bmatrix}_U\begin{Bmatrix}6n+3\\2n\end{Bmatrix}_U,$$

17-o)

$$\sum_{k=0}^{4n+3}\begin{Bmatrix}4n+3\\k\end{Bmatrix}_U^2\begin{Bmatrix}4n+5\\k\end{Bmatrix}_U(-1)^{\frac{1}{2}k(k-1)}$$
$$=(-1)^{n+1}V_{2n+3}\begin{Bmatrix}4n+3\\2n+1\end{Bmatrix}_U\begin{Bmatrix}6n+6\\2n+1\end{Bmatrix}_U.$$

18-e)

$$\sum_{k=0}^{4n+1}\begin{Bmatrix}4n+1\\k\end{Bmatrix}_U^2\begin{Bmatrix}4n+3\\k+1\end{Bmatrix}_U U_{2k}\,(-1)^{\frac{1}{2}k(k-1)}$$
$$=(-1)^n\sqrt{\Delta}U_{4n+1}U_{4n+3}\begin{Bmatrix}4n+1\\2n\end{Bmatrix}_U\begin{Bmatrix}6n+2\\2n\end{Bmatrix}_U,$$

18-o)

$$\sum_{k=0}^{4n+3}\begin{Bmatrix}4n+3\\k\end{Bmatrix}_U^2\begin{Bmatrix}4n+5\\k+1\end{Bmatrix}_U U_{2k}\,(-1)^{\frac{1}{2}k(k+1)}$$
$$=(-1)^{n+1}\sqrt{\Delta}U_{4n+3}U_{4n+5}\begin{Bmatrix}4n+3\\2n+1\end{Bmatrix}_U\begin{Bmatrix}6n+5\\2n+1\end{Bmatrix}_U.$$

19-e)

$$\sum_{k=0}^{4n+1}\begin{Bmatrix}4n+1\\k\end{Bmatrix}_U^2\begin{Bmatrix}4n+4\\k+2\end{Bmatrix}_U U_k(-1)^{\frac{1}{2}k(k+1)}$$
$$=(-1)^nV_1\frac{U_{4n+4}}{U_{2n}}\begin{Bmatrix}4n+1\\2n-1\end{Bmatrix}_U\begin{Bmatrix}6n+3\\2n\end{Bmatrix}_U,$$

19-o)

$$\sum_{k=0}^{4n+3}\begin{Bmatrix}4n+3\\k\end{Bmatrix}_U^2\begin{Bmatrix}4n+6\\k+2\end{Bmatrix}_U U_k(-1)^{\frac{1}{2}k(k-1)}$$
$$=(-1)^{n+1}V_1\frac{U_{4n+6}}{U_{2n+1}}\begin{Bmatrix}4n+3\\2n\end{Bmatrix}_U\begin{Bmatrix}6n+6\\2n+1\end{Bmatrix}_U.$$

20-e)

$$\sum_{k=0}^{4n+1}\begin{Bmatrix}4n+1\\k\end{Bmatrix}_U^2\begin{Bmatrix}4n+4\\k+2\end{Bmatrix}_U V_k(-1)^{\frac{1}{2}k(k+1)}$$
$$=(-1)^nV_2\frac{U_{4n+4}}{U_{2n}}\begin{Bmatrix}4n+1\\2n-1\end{Bmatrix}_U\begin{Bmatrix}6n+3\\2n\end{Bmatrix}_U,$$

20-o)

$$\sum_{k=0}^{4n+3}\begin{Bmatrix}4n+3\\k\end{Bmatrix}_U^2\begin{Bmatrix}4n+6\\k+2\end{Bmatrix}_U V_k(-1)^{\frac{1}{2}k(k-1)}$$
$$=(-1)^{n+1}V_2\frac{U_{4n+6}}{U_{2n+1}}\begin{Bmatrix}4n+3\\2n\end{Bmatrix}_U\begin{Bmatrix}6n+6\\2n+1\end{Bmatrix}_U.$$

21-e)

$$\sum_{k=0}^{4n+1} \left\{ {4n+1 \atop k} \right\}_U^2 \left\{ {4n+4 \atop k+1} \right\}_U U_k^3 \, (-1)^{\frac{1}{2}k(k+1)} = \\ (-1)^n \frac{U_{4n+1}U_{4n+3}U_{4n+4}U_{2n+1}}{U_{2n}} \left\{ {4n+1 \atop 2n-1} \right\}_U \left\{ {6n+2 \atop 2n} \right\}_U,$$

21-o)

$$\sum_{k=0}^{4n+3} \left\{ {4n+3 \atop k} \right\}_U^2 \left\{ {4n+6 \atop k+1} \right\}_U U_k^3 \, (-1)^{\frac{1}{2}k(k-1)} = \\ (-1)^{n+1} \frac{U_{4n+3}U_{4n+5}U_{4n+6}U_{2n+2}}{U_{2n+1}} \left\{ {4n+3 \atop 2n} \right\}_U \left\{ {6n+5 \atop 2n+1} \right\}_U.$$

22-e)

$$\sum_{k=0}^{4n+1} \left\{ {4n+1 \atop k} \right\}_U \left\{ {4n+3 \atop k} \right\}_U^2 U_k^2 (-1)^{\frac{1}{2}k(k-1)} \\ = \Delta^{-1/2}(-1)^n \frac{U_{2n+2}}{U_{2n+1}} \left\{ {4n+3 \atop 2n+1} \right\}_U \left\{ {6n+3 \atop 2n+1} \right\}_U,$$

22-o)

$$\sum_{k=0}^{4n+3} \left\{ {4n+3 \atop k} \right\}_U \left\{ {4n+5 \atop k} \right\}_U^2 U_k^2 (-1)^{\frac{1}{2}k(k+1)} \\ = \Delta^{-1/2}(-1)^{n-1} \frac{U_{2n+3}}{U_{2n+2}} \left\{ {4n+5 \atop 2n+2} \right\}_U \left\{ {6n+6 \atop 2n+2} \right\}_U.$$

23-e)

$$\sum_{k=0}^{4n+1} \left\{ {4n+1 \atop k} \right\}_U \left\{ {4n+3 \atop k} \right\}_U^2 (-1)^{\frac{1}{2}k(k+1)} = (-1)^{n-1} \left\{ {4n+2 \atop 2n+1} \right\}_U \left\{ {6n+4 \atop 2n+1} \right\}_U,$$

23-o)

$$\sum_{k=0}^{4n+3} \left\{ {4n+3 \atop k} \right\}_U \left\{ {4n+5 \atop k} \right\}_U^2 (-1)^{\frac{1}{2}k(k-1)} = (-1)^n \left\{ {4n+4 \atop 2n+2} \right\}_U \left\{ {6n+7 \atop 2n+2} \right\}_U.$$

24-e)

$$\sum_{k=0}^{4n+1} \left\{ {4n+1 \atop k} \right\}_U^2 \left\{ {4n+5 \atop k+1} \right\}_U (-1)^{\frac{1}{2}k(k-1)} \\ = (-1)^n \frac{U_{4n+5}U_{4n+6}}{U_{2n-2}U_{2n}} \left\{ {4n+1 \atop 2n-2} \right\}_U \left\{ {6n+4 \atop 2n} \right\}_U,$$

24-o)

$$\sum_{k=0}^{4n+3} \begin{Bmatrix} 4n+3 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+7 \\ k+1 \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k+1)} = (-1)^{n+1} \frac{U_{4n+5}U_{4n+6}}{U_{2n}U_{2n+1}} \begin{Bmatrix} 4n+3 \\ 2n-1 \end{Bmatrix}_U \begin{Bmatrix} 6n+7 \\ 2n+1 \end{Bmatrix}_U.$$

25-e)

$$\sum_{k=0}^{4n+1} \begin{Bmatrix} 4n+1 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+5 \\ k+1 \end{Bmatrix}_U U_k^2 (-1)^{\frac{1}{2}k(k+1)} = (-1)^{n} \frac{U_{4n+1}U_{4n+4}U_{4n+5}}{U_{2n}} \begin{Bmatrix} 4n+1 \\ 2n-1 \end{Bmatrix}_U \begin{Bmatrix} 6n+3 \\ 2n \end{Bmatrix}_U,$$

25-o)

$$\sum_{k=0}^{4n+3} \begin{Bmatrix} 4n+3 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+7 \\ k+1 \end{Bmatrix}_U U_k^2 (-1)^{\frac{1}{2}k(k-1)} = (-1)^{n-1} \frac{U_{4n+3}U_{4n+6}U_{4n+7}}{U_{2n+1}} \begin{Bmatrix} 4n+3 \\ 2n \end{Bmatrix}_U \begin{Bmatrix} 6n+6 \\ 2n+1 \end{Bmatrix}_U,$$

where $\Delta = p^2 + 4$.

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