FORMULÆ RELATED TO THE q-DIXON FORMULA WITH APPLICATIONS TO FIBONOMIAL SUMS

EMRAH KILIÇ AND HELMUT PRODINGER

ABSTRACT. The q-analogue of Dixon's identity involves three q-binomial coefficients as summands. We find many variations of it that have beautiful corollories in terms of Fibonomial sums. Proofs involve either several instances of the q-Dixon formula itself or are "mechanical," i. e., use the q-Zeilberger algorithm.

1. INTRODUCTION

Define the second order linear sequence $\{U_n\}$ for $n \geq 2$ by

$$U_n = pU_{n-1} + U_{n-2}, \quad U_0 = 0,\ U_1 = 1.$$

For $n \geq k \geq 1$, define the generalized Fibonomial coefficient by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_U := \frac{U_1 U_2 \dots U_n}{(U_1 U_2 \dots U_k)(U_1 U_2 \dots U_{n-k})}$$

with $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\}_U = \left\{ \begin{smallmatrix} n \\ n \end{smallmatrix} \right\}_U = 1$. When $p = 1$, we obtain the usual Fibonomial coefficient, denoted by $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_F$. For more details about the Fibonomial and generalized Fibonomial coefficients, see [2, 3].

Our approach will be as follows. We will use the Binet forms

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{1 - q^n}{1 - q}$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$ where $\alpha, \beta = \left(p \pm \sqrt{p^2 + 4}\right)/2$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x; q)_n = (1 - x)(1 - xq) \dots (1 - xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.$$

When $x = q$, we sometimes use the notation $(q)_n$ instead of $(q; q)_n$. We conveniently adopt the notation that $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q = 0$ if $k < 0$ or $k > n$.

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_U = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \text{with} \quad q = -\alpha^{-2}.$$

We recall the q-analogue of Dixon's identity [1, 4], which is central in this paper:

$$\sum_k (-1)^k q^{\frac{k}{2}(3k+1)} \begin{bmatrix} a+b \\ a+k \end{bmatrix}_q \begin{bmatrix} b+c \\ b+k \end{bmatrix}_q \begin{bmatrix} c+a \\ c+k \end{bmatrix}_q = \frac{[a+b+c]!}{[a]![b]![c]!},$$

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where $[n]! = \prod_{i=1}^{n} \frac{1-q^i}{1-q} = (q;q)_n/(1-q)^n$.

Recently the authors of [5, 6] proved sum identities including certain generalized Fibonomial sums and their squares with or without the generalized Fibonacci and Lucas numbers. We recall such a result: if n and m are both *nonnegative* integers, then from [5], we have that

$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\}_U U_{(2m-1)k} = T_{n,m} \sum_{k=1}^{m} \left\{ {2m-1 \atop 2k-1} \right\}_U U_{(4k-2)n},$$

where

$$T_{n,m} = \begin{cases} \prod\limits_{k=0}^{n-m} V_{2k} & \text{if } \ n \geq m, \\ \prod\limits_{k=1}^{m-n-1} V_{2k}^{-1} & \text{if } \ n < m, \end{cases}$$

and three similar formulæ.

From [6], we have that for any positive integer n,

$$\sum_{k=0}^{2n} \mathbf{i}^{\pm k} \left\{ {2n \atop k} \right\}_U = \mathbf{i}^{\pm n} \prod_{k=1}^{n} V_{2k-1},$$

$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\}_U^2 = \prod_{k=1}^{n} \frac{V_{2k} U_{2(2k-1)}}{U_{2k}}$$

and

$$\sum_{k=0}^{n} (-1)^k \left\{ {2n+1 \atop 2k+1} \right\}_U = (-1)^{\binom{n}{2}} \begin{cases} \prod\limits_{k=1}^{n} V_k^2 & \text{if } n \text{ is odd}, \\ \prod\limits_{k=1}^{n} V_{2k} & \text{if } n \text{ is even}. \end{cases}$$

In this paper, we consider some sum formulæ whose terms include certain triple Fibonomial coefficients, with or without extra Fibonacci numbers. To be systematic, we first organize the q-Dixon type identities in a list (a much longer list is in [7]), then discuss the proofs of them, and then get a list of Fibonacci type identities as corollaries.

2. Triple Gaussian q-Binomial Sums

In this section, we present some sum formulæ. In order to keep this paper within reasonable length, we restricted ourselves to a short selection. We prepared an extended version of this paper with all identities we found and put in on our websites for the readers' benefit [7]. The identities in this section hold for all nonnegative integers n.

(1)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-1)} = (-1)^n q^{-\frac{n}{2}(3n+1)} \begin{bmatrix} 2n \\ n \end{bmatrix}_q \begin{bmatrix} 3n+1 \\ n \end{bmatrix}_q.$$

(2)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)}(1+q^{2k}) = 2(-1)^n q^{-\frac{n}{2}(3n+1)} \begin{bmatrix} 2n \\ n \end{bmatrix}_q \begin{bmatrix} 3n+1 \\ n \end{bmatrix}_q.$$

(3)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)}(1-q^{2k}) = 2(-1)^n q^{-\frac{n}{2}(3n+1)}(1-q^{2n+1}) \begin{bmatrix} 2n \\ n \end{bmatrix}_q \begin{bmatrix} 3n \\ n-1 \end{bmatrix}_q.$$

(4)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+2 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-5)} \left(1-q^k\right) = (-1)^{n+1} q^{-\frac{1}{2}(n+1)(3n+2)} \left(1-q^{2n+2}\right) \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q \begin{bmatrix} 3n+2 \\ n \end{bmatrix}_q.$$

(5)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n+2 \\ k \end{bmatrix}_q^2 (-1)^k q^{\frac{k}{2}(3k-6n-5)} = (-1)^{n+1} q^{-\frac{1}{2}(n+1)(3n+2)} \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q \begin{bmatrix} 3n+3 \\ n+1 \end{bmatrix}_q.$$

(6)

$$\sum_{k=0}^{2n+1} \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n+2 \\ k \end{bmatrix}_q^2 (-1)^k q^{\frac{k}{2}(3k-6n-7)} \left(1-q^k\right)^2 = (-1)^{n+1} q^{-\frac{1}{2}(n+1)(3n+4)} \left(1-q^{2n+2}\right)^2 \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q \begin{bmatrix} 3n+2 \\ n \end{bmatrix}_q.$$

(7)

$$\sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+3 \\ k+1 \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-1)} = (-1)^n q^{-\frac{n}{2}(3n+1)} \frac{1-q^{2n+3}}{1-q^n} \begin{bmatrix} 2n \\ n-1 \end{bmatrix}_q \begin{bmatrix} 3n+2 \\ n \end{bmatrix}_q.$$

3. Proofs

In this section we prove the identities 1, 2, 3, 4, 5, 6 using the q-Dixon formula.
Proof of identity 1.

First if we replace $k \to n-k$, then we write

$$\sum_k \begin{bmatrix} 2n \\ n-k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ n-k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k+1)} = \begin{bmatrix} 2n \\ n \end{bmatrix}_q \begin{bmatrix} 3n+1 \\ n \end{bmatrix}_q,$$

which is an equivalent form of identity (1). Another equivalent form is

$$\sum_k (1-q^{2n+1}) \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k+1)} = \frac{(q)_{3n+1}}{(q)_n^3},$$

and this one we will prove now by two applications of Dixon's formula. Note that within the following computations, we sometimes change $k \leftrightarrow -k$ in order to transform the exponent $\frac{k(3k-1)}{2}$ to $\frac{k(3k+1)}{2}$.

$$\begin{aligned}
&\sum_k (1-q^{2n+1}) \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k+1)} \\
&\quad= \sum_k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q (1-q^{n+1-k})(-1)^k q^{\frac{k}{2}(3k+1)} \\
&\quad= \frac{(q)_{3n+1}}{(q)_n^2 (q)_{n+1}} - \sum_k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q q^{n+1-k}(-1)^k q^{\frac{k}{2}(3k+1)} \\
&\quad= \frac{(q)_{3n+1}}{(q)_n^2 (q)_{n+1}} - q^{n+1} \sum_k \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k+1)} \\
&\quad= \frac{(q)_{3n+1}}{(q)_n^2 (q)_{n+1}} - q^{n+1} \frac{(q)_{3n+1}}{(q)_n^2 (q)_{n+1}} \\
&\quad= \frac{(q)_{3n+1}}{(q)_n^3}.
\end{aligned}$$

Proof of identity 5. By taking $k \to n+1-k$ and after some rearrangements, then we write

$$\sum_k (1-q^{2n+2}) \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_q^2 (-1)^k q^{\frac{k}{2}(3k-1)} = \frac{(q)_{3n+3}}{(q)_n (q)_{n+1}^2}.$$

This form is equivalent to identity (5) and will be proved now by two applications of Dixon's identity.

$$\begin{aligned}
&\sum_k (1-q^{2n+2}) \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_q^2 (-1)^k q^{\frac{k}{2}(3k-1)} \\
&\quad= \sum_k (1-q^{n+1+k}) \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_q^3 (-1)^k q^{\frac{k}{2}(3k-1)} \\
&\quad= \frac{(q)_{3n+3}}{(q)_{n+1}^3} - q^{n+1} \sum_k \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_q^3 (-1)^k q^{\frac{k}{2}(3k+1)} \\
&\quad= \frac{(q)_{3n+3}}{(q)_{n+1}^3} - q^{n+1} \frac{(q)_{3n+3}}{(q)_{n+1}^3} \\
&\quad= \frac{(q)_{3n+3}}{(q)_n (q)_{n+1}^2}.
\end{aligned}$$

Proof of identity 4. By replacing $k \to n+1+k$ and rearrangements, we get the equivalent form

$$\sum_k \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q \times (-1)^k q^{\frac{k}{2}(3k+1)}(1-q^{n+1-k}) = \frac{(q)_{3n+2}}{(q)_n^2 (q)_{n+1}}.$$

It will be proved by two applications of Dixon's formula:

$$\begin{aligned}
&\sum_k \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k+1)}(1-q^{n+1-k}) \\
&= \frac{(q)_{3n+2}}{(q)_n (q)_{n+1}^2} - q^{n+1} \sum_k \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-1)} \\
&= \frac{(q)_{3n+2}}{(q)_n (q)_{n+1}^2} - q^{n+1} \frac{(q)_{3n+2}}{(q)_n (q)_{n+1}^2} \\
&= \frac{(q)_{3n+2}}{(q)_n^2 (q)_{n+1}}.
\end{aligned}$$

Proof of identity 6.

By taking $k \to n+1-k$ and some rearrangements, the claimed identity takes the equivalent form

$$\sum_k (1-q^{2n+2}) \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q^2 (-1)^k q^{\frac{k}{2}(3k+1)} = \frac{(q)_{3n+2}}{(q)_n^2 (q)_{n+1}},$$

which will be proved by Dixon's formula:

$$\begin{aligned}
&\sum_k (1-q^{2n+2}) \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q^2 (-1)^k q^{\frac{k}{2}(3k+1)} \\
&= \sum_k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q (-1)^k (1-q^{n+1-k}) q^{\frac{k}{2}(3k+1)} \\
&= \frac{(q)_{3n+2}}{(q)_n (q)_{n+1}^2} - q^{n+1} \sum_k \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-1)} \\
&= \frac{(q)_{3n+2}}{(q)_n (q)_{n+1}^2} - q^{n+1} \sum_k \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k+1)} \\
&= \frac{(q)_{3n+2}}{(q)_n (q)_{n+1}^2} - q^{n+1} \frac{(q)_{3n+2}}{(q)_n (q)_{n+1}^2} \\
&= \frac{(q)_{3n+2}}{(q)_n^2 (q)_{n+1}}.
\end{aligned}$$

Proof of identity 3. This proof is more involved and requires auxiliary quantities that will be evaluated by several applications of Dixon's identity. Define

$$T := \sum_k \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} q^k,$$

$$W := \sum_k \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} q^{2k}$$

and

$$X := \sum_k \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)}.$$

To complete the proof we should prove that

$$X - W = 2(-1)^n q^{-\frac{n}{2}(3n+1)} (1 - q^{2n+1}) \begin{bmatrix} 2n \\ n \end{bmatrix}_q \begin{bmatrix} 3n \\ n-1 \end{bmatrix}_q.$$

First we notice that T is the sum in identity (1), so

$$T = (-1)^n q^{-\frac{n}{2}(3n+1)} \frac{1}{1-q^{2n+1}} \frac{(q)_{3n+1}}{(q)_n (q)_n (q)_n}.$$

Next we compute

$$\begin{aligned}
V &= \sum_k \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} (1-q^k)^2 \\
&= (1-q^{2n})(1-q^{2n+1}) \sum_k (-1)^k q^{\frac{k}{2}(3k-6n-3)} \begin{bmatrix} 2n-1 \\ k-1 \end{bmatrix}_q \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n \\ k-1 \end{bmatrix}_q \\
&= (1-q^{2n})(1-q^{2n+1}) \sum_k (-1)^k q^{\frac{k}{2}(3k-6n-3)} \begin{bmatrix} 2n-1 \\ 2n-k \end{bmatrix}_q \begin{bmatrix} 2n \\ 2n-k \end{bmatrix}_q \begin{bmatrix} 2n \\ 2n+1-k \end{bmatrix}_q \\
&= (1-q^{2n})(1-q^{2n+1}) \sum_j (-1)^{j-1} q^{\frac{j}{2}(3j-6n-3)} \begin{bmatrix} 2n-1 \\ j-1 \end{bmatrix}_q \begin{bmatrix} 2n \\ j \end{bmatrix}_q \begin{bmatrix} 2n \\ j-1 \end{bmatrix}_q \\
&= -V,
\end{aligned}$$

hence $V = 0$. Therefore we get

$$\begin{aligned}
&\sum_k \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} (1-q^k) \\
&\quad = \sum_k \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q (-1)^k q^{\frac{k}{2}(3k-6n-3)} (1-q^k) q^k
\end{aligned}$$

and thus

$$X - T = T - W$$

and so

$$X + W = 2T,$$

which will be used later. Now we compute

$$\begin{aligned}
W &= \sum_k (-1)^k q^{\frac{k}{2}(3k-6n-3)} q^{2k} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q \\
&= (-1)^n q^{-\frac{n}{2}(3n-1)} \sum_k (-1)^k q^{\frac{k}{2}(3k+1)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q
\end{aligned}$$

$$= (-1)^n q^{-\frac{n}{2}(3n-1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k+1)} (1-q^{n+1+k}) \times \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$
$$= (-1)^n q^{-\frac{n}{2}(3n-1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k+1)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$
$$- (-1)^n q^{-\frac{n}{2}(3n-1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k+1)} q^{n+1+k} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$
$$= (-1)^n q^{-\frac{n}{2}(3n-1)} \frac{1}{1-q^{2n+1}} \frac{(q)_{3n+1}}{(q)_n (q)_n (q)_{n+1}}$$
$$- (-1)^n q^{-\frac{n}{2}(3n-1)+\frac{n}{2}+1} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k+3)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$

and

$$X = \sum_k (-1)^k q^{\frac{k}{2}(3k-6n-3)} \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n \\ k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ k \end{bmatrix}_q$$
$$= (-1)^n q^{-\frac{3}{2}n(n+1)} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$
$$= (-1)^n \frac{q^{-\frac{3}{2}n(n+1)}}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} (1-q^{n+k+1}) \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$
$$= (-1)^n q^{-\frac{3}{2}n(n+1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$
$$- (-1)^n q^{-\frac{3}{2}n(n+1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} q^{n+k+1} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$
$$= (-1)^n q^{-\frac{3}{2}n(n+1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$
$$- (-1)^n q^{-\frac{1}{2}(3n-2)(n+1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-1)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q,$$

which by $k \to -k$ in the second sum, equals

$$= (-1)^n q^{-\frac{3}{2}n(n+1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$
$$- (-1)^n q^{-\frac{1}{2}(3n-2)(n+1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k+1)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+1+k \end{bmatrix}_q$$
$$= (-1)^n q^{-\frac{3}{2}n(n+1)} \frac{1}{1-q^{2n+1}} \sum_k (-1)^k q^{\frac{k}{2}(3k-3)} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k+1 \end{bmatrix}_q \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_q$$
$$- (-1)^n q^{-\frac{1}{2}(3n-2)(n+1)} \frac{1}{1-q^{2n+1}} \frac{(q)_{3n+1}}{(q)_n (q)_n (q)_{n+1}}.$$

Consequently we have the summarized results

$$\begin{aligned}W&=(-1)^nq^{-\frac{n}{2}(3n-1)}\frac{1}{1-q^{2n+1}}\frac{(q)_{3n+1}}{(q)_n(q)_n(q)_{n+1}}\\&\quad-(-1)^nq^{-\frac{n}{2}(3n-1)+n+1}\frac{1}{1-q^{2n+1}}\sum_k(-1)^kq^{\frac{k}{2}(3k-3)}\begin{bmatrix}2n\\n+k\end{bmatrix}_q\begin{bmatrix}2n+1\\n+k\end{bmatrix}_q\begin{bmatrix}2n+1\\n+1+k\end{bmatrix}_q\end{aligned}$$

and

$$\begin{aligned}X&=(-1)^nq^{-\frac{3}{2}n(n+1)}\frac{1}{1-q^{2n+1}}\sum_k(-1)^kq^{\frac{k}{2}(3k-3)}\begin{bmatrix}2n\\n+k\end{bmatrix}_q\begin{bmatrix}2n+1\\n+k+1\end{bmatrix}_q\begin{bmatrix}2n+1\\n+k\end{bmatrix}_q\\&\quad-(-1)^nq^{-\frac{1}{2}(3n-1)(n+1)}\frac{1}{1-q^{2n+1}}\frac{(q)_{3n+1}}{(q)_n(q)_n(q)_{n+1}}.\end{aligned}$$

Therefore

$$\begin{aligned}&q^{3n+1}X+W\\&\quad=-(-1)^nq^{-\frac{1}{2}(3n-2)(n+1)}q^{3n+1}\frac{1}{1-q^{2n+1}}\frac{(q)_{3n+1}}{(q)_n(q)_n(q)_{n+1}}\\&\qquad\quad+(-1)^nq^{-\frac{n}{2}(3n-1)}\frac{1}{1-q^{2n+1}}\frac{(q)_{3n+1}}{(q)_n(q)_n(q)_{n+1}}\\&\quad=(-1)^nq^{-\frac{n}{2}(3n-1)}\frac{1-q^{2n+2}}{1-q^{2n+1}}\frac{(q)_{3n+1}}{(q)_n(q)_n(q)_{n+1}}.\end{aligned}$$

We can rewrite this as

$$q^{3n+1}X+W=T(1+q^{n+1})q^n.$$

But we also know that

$$W+X=2T.$$

From these two relations, we can compute X and W and thus $X-W$ as

$$\begin{aligned}X&=T\frac{1}{1-q^{3n+1}}\Big(2-(1+q^{n+1})q^n\Big)\\&=(-1)^nq^{-\frac{n}{2}(3n+1)}\Big(2-(1+q^{n+1})q^n\Big)\frac{1}{1-q^{2n+1}}\frac{(q)_{3n}}{(q)_n(q)_n(q)_n}\end{aligned}$$

and

$$\begin{aligned}W&=2T-X\\&=(-1)^nq^{-\frac{n}{2}(3n+1)+n}(1+q^{n+1}-2q^{2n+1})\frac{1}{1-q^{3n+1}}\frac{1}{1-q^{2n+1}}\frac{(q)_{3n+1}}{(q)_n(q)_n(q)_n},\end{aligned}$$

and so the result

$$\begin{aligned}X-W&=T\frac{1}{1-q^{3n+1}}\Big(2-(1+q^{n+1})q^n\Big)-T\frac{1}{1-q^{3n+1}}q^n(1+q^{n+1}-2q^{2n+1})\\&=T\frac{1}{1-q^{3n+1}}\Big((2-(1+q^{n+1})q^n)-q^n(1+q^{n+1}-2q^{2n+1})\Big)\\&=2T(1-q^n)(1-q^{2n+1})\frac{1}{1-q^{3n+1}}\\&=2(-1)^nq^{-\frac{n}{2}(3n+1)}(1-q^n)(1-q^{2n+1})\frac{1}{1-q^{2n+1}}\frac{(q)_{3n}}{(q)_n(q)_n(q)_n}\end{aligned}$$

$$= 2(-1)^n q^{-\frac{n}{2}(3n+1)} \frac{(q)_{3n}}{(q)_n (q)_n (q)_{n-1}},$$

as claimed.

Remark. From this proof we know that

$$X + W = 2T,$$

which proves the identity 2.

As the last example has shown, the reduction to instances of the q-Dixon identity can be quite involved. Therefore we present an alternative method, namely the q-Zeilberger algorithm [8]. We discuss identity 7 as a showcase: Define

$$T_n := \sum_{k=0}^{2n} \begin{bmatrix} 2n \\ k \end{bmatrix}_q^2 \begin{bmatrix} 2n+3 \\ k+1 \end{bmatrix}_q (-1)^k \, q^{\frac{k}{2}(3k-6n-1)}.$$

Zeilberger's algorithm produces a recursion

$$a_n T_n + b_n T_{n+1} + c_n T_{n+2} + d_n T_{n+3} = 0,$$

where a_n, b_n, c_n, d_n are complicated expressions with about 1000 terms each.

Set

$$U_n := (-1)^n q^{-\frac{n}{2}(3n+1)} \frac{1-q^{2n+3}}{1-q^n} \begin{bmatrix} 2n \\ n-1 \end{bmatrix}_q \begin{bmatrix} 3n+2 \\ n \end{bmatrix}_q,$$

then it can be checked (by a computer) that also

$$a_n U_n + b_n U_{n+1} + c_n U_{n+2} + d_n U_{n+3} = 0.$$

After checking a few initial values directly, this proves indeed that $T_n = U_n$ for all nonnegative integers n.

4. Applications to Fibonomials Sums Identities

In this section, we present corollaries of our previous list of identities, by specializing the value of q as described in the Introduction. Each identity corresponds now to two identities which have slightly different forms. By replacing $n \to 2n$, we get a formula labelled with "e" (even), and by replacing $n \to 2n+1$, we get a formula labelled with "o" (odd).

1-e)

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+3 \\ k \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k+1)} = (-1)^{n+1} \begin{Bmatrix} 4n+2 \\ 2n+1 \end{Bmatrix}_U \begin{Bmatrix} 6n+4 \\ 2n+1 \end{Bmatrix}_U.$$

1-o)

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+1 \\ k \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k-1)} = (-1)^n \begin{Bmatrix} 4n \\ 2n \end{Bmatrix}_U \begin{Bmatrix} 6n+1 \\ 2n \end{Bmatrix}_U,$$

2-e)

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+1 \\ k \end{Bmatrix}_U V_{2k} (-1)^{\frac{1}{2}k(k+1)} = 2(-1)^n \begin{Bmatrix} 4n \\ 2n \end{Bmatrix}_U \begin{Bmatrix} 6n+1 \\ 2n \end{Bmatrix}_U,$$

2-o)

$$\sum_{k=0}^{4n+2} \left\{ {4n+2 \atop k} \right\}_U^2 \left\{ {4n+3 \atop k} \right\}_U V_{2k} (-1)^{\frac{1}{2}k(k-1)} = 2(-1)^{n+1} \left\{ {4n+2 \atop 2n+1} \right\}_U \left\{ {6n+4 \atop 2n+1} \right\}_U .$$

3-e)

$$\sum_{k=0}^{4n} \left\{ {4n \atop k} \right\}_U^2 \left\{ {4n+1 \atop k} \right\}_U U_{2k} (-1)^{\frac{k}{2}(k+1)} = 2(-1)^n U_{4n+1} \left\{ {4n \atop 2n} \right\}_U \left\{ {6n \atop 2n-1} \right\}_U ,$$

3-o)

$$\sum_{k=0}^{4n+2} \left\{ {4n+2 \atop k} \right\}_U^2 \left\{ {4n+3 \atop k} \right\}_U U_{2k} (-1)^{\frac{1}{2}k(k-1)}$$
$$= 2(-1)^{n+1} U_{4n+3} \left\{ {4n+2 \atop 2n+1} \right\}_U \left\{ {6n+3 \atop 2n} \right\}_U .$$

4-e)

$$\sum_{k=0}^{4n+1} \left\{ {4n+1 \atop k} \right\}_U^2 \left\{ {4n+2 \atop k} \right\}_U U_k \, (-1)^{\frac{1}{2}k(k-1)}$$
$$= (-1)^n U_{4n+2} \left\{ {4n+1 \atop 2n} \right\}_U \left\{ {6n+2 \atop 2n} \right\}_U ,$$

4-o)

$$\sum_{k=0}^{4n+3} \left\{ {4n+3 \atop k} \right\}_U^2 \left\{ {4n+4 \atop k} \right\}_U U_k \, (-1)^{\frac{1}{2}k(k+1)}$$
$$= (-1)^{n+1} \, U_{4n+4} \left\{ {4n+3 \atop 2n+1} \right\}_U \left\{ {6n+5 \atop 2n+1} \right\}_U .$$

5-e)

$$\sum_{k=0}^{4n+1} \left\{ {4n+1 \atop k} \right\}_U \left\{ {4n+2 \atop k} \right\}_U^2 (-1)^{\frac{1}{2}k(k-1)} = (-1)^n \left\{ {4n+1 \atop 2n} \right\}_U \left\{ {6n+3 \atop 2n+1} \right\}_U ,$$

5-o)

$$\sum_{k=0}^{4n+3} \left\{ {4n+3 \atop k} \right\}_U \left\{ {4n+4 \atop k} \right\}_U^2 (-1)^{\frac{1}{2}k(k+1)} = (-1)^{n+1} \left\{ {4n+3 \atop 2n+1} \right\}_U \left\{ {6n+6 \atop 2n+2} \right\}_U .$$

6-e)

$$\sum_{k=0}^{4n+1} \left\{ {4n+1 \atop k} \right\}_U \left\{ {4n+2 \atop k} \right\}_U^2 U_k^2 \, (-1)^{\frac{1}{2}k(k+1)}$$
$$= (-1)^n U_{4n+2}^2 \left\{ {4n+1 \atop 2n} \right\}_U \left\{ {6n+2 \atop 2n} \right\}_U ,$$

6-o)

$$\sum_{k=0}^{4n+3} \begin{Bmatrix} 4n+3 \\ k \end{Bmatrix}_U \begin{Bmatrix} 4n+4 \\ k \end{Bmatrix}_U^2 U_k^2 (-1)^{\frac{1}{2}k(k-1)} = (-1)^{n+1} U_{4n+4}^2 \begin{Bmatrix} 4n+3 \\ 2n+1 \end{Bmatrix}_U \begin{Bmatrix} 6n+5 \\ 2n+1 \end{Bmatrix}_U.$$

7-e)

$$\sum_{k=0}^{4n} \begin{Bmatrix} 4n \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+3 \\ k+1 \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k-1)} = (-1)^n \frac{U_{4n+3}}{U_{2n}} \begin{Bmatrix} 4n \\ 2n-1 \end{Bmatrix}_U \begin{Bmatrix} 6n+2 \\ 2n \end{Bmatrix}_U,$$

7-o)

$$\sum_{k=0}^{4n+2} \begin{Bmatrix} 4n+2 \\ k \end{Bmatrix}_U^2 \begin{Bmatrix} 4n+5 \\ k+1 \end{Bmatrix}_U (-1)^{\frac{1}{2}k(k+1)} = (-1)^{n+1} \frac{U_{4n+5}}{U_{2n+1}} \begin{Bmatrix} 4n+2 \\ 2n \end{Bmatrix}_U \begin{Bmatrix} 6n+5 \\ 2n+1 \end{Bmatrix}_U.$$

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TOBB University of Economics and Technology Mathematics Department 06560 Ankara Turkey

E-mail address: ekilic@etu.edu.tr

Department of Mathematics, University of Stellenbosch 7602 Stellenbosch South Africa

E-mail address: hproding@sun.ac.za