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VARIOUS SUMS INCLUDING THE GENERALIZED FIBONACCI AND LUCAS NUMBERS

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Abstract We compute certain sums including generalized Fibonacci and Lucas numbers as well as their alternating analogous. We show that these sums could be expressed by the terms of the sequences. Similar directions are given for certain sums of product of generalized Fibonacci and Lucas numbers with their alternating analogous.

1 Introduction

Let p be a nonzero integer such that $\Delta = p^2 + 4 \neq 0$. The generalized Fibonacci and Lucas numbers are defined by the recursions: for $n > 0$

$$\begin{aligned} U_{n+1} &= pU_n + U_{n-1}, \\ V_{n+1} &= pV_n + V_{n-1}, \end{aligned}$$

where $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = p$, respectively.

The Binet formulas are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n,$$

where α and β are the roots of the equation $x^2 - px - 1 = 0$.

In this paper, we consider certain generalized Fibonacci and Lucas sums as well as their alternating analogous whose lower bounds are nonnegative integer variables. As an earlier first examples of such sums, we could refer to [5, 6] and recall two examples here:

$$\sum_{k=a+1}^{a+4n} F_k = F_{2n} L_{a+2n+2} \quad \text{and} \quad \sum_{k=n}^{n+9} F_k = 11 F_{n+6}.$$

Many authors have studied various interesting Fibonacci and Lucas sums (see [2, 3, 4]). For example, we recall the results from [1, 4]:

$$\sum_{k=1}^{2n+1} F_k F_{k-1} = F_{2n}^2 \quad \text{and} \quad \sum_{k=1}^{2n+1} L_k L_{k-1} = L_{2n}^2 - 4. \tag{1.1}$$

In this paper, for instance, our approach to generalize the identities given in (1.1) would be as follows

$$\sum_{i=n}^{n+2k+1} U_{i+a} U_{i+b} = \frac{U_{2(k+1)} U_{2(n+k)+a+b+1}}{U_2}$$

and

$$\sum_{i=n}^{n+2k+1} V_{(2m+1)i+a} V_{(2m+1)i+b} = \frac{1}{V_{2m+1}} \times U_{2(k+1)(2m+1)} \left[V_{2(2m+1)(n+k+1)-2m+a+b} + V_{2(2m+1)(n+k+1)-2(m+1)+a+b} \right],$$

respectively. Furthermore, we consider certain similar sums including product of generalized Fibonacci and Lucas numbers as well as their alternating analogous. We show that many of these sums could be written in terms product of Fibonacci and Lucas numbers.

2 Sums including the generalized Fibonacci and Lucas numbers

Theorem 2.1. *For $k \geq 0$, and, any integers n, m and a,*

$$\sum_{i=n}^{n+2k+1} U_{i+a} = \frac{1}{U_2}\begin{cases} U_{k+1}V_{n+k+a+1} - U_{n+a-1} + U_{n+2k+a+1} & \text{if } k \text{ is odd,} \\ V_{k+1}U_{n+k+a+1} - U_{n+a-1} + U_{n+2k+a+1} & \text{if } k \text{ is even,} \end{cases}$$

$$\sum_{i=n}^{n+2k+1} U_{2mi+a} = \begin{cases} U_{2m(k+1)}V_{2m(n+k+1)+a-m}/V_m & \text{if } m \text{ is odd,} \\ U_{2m(k+1)}U_{2m(n+k+1)+a-m}/U_m & \text{if } m \text{ is even,} \end{cases}$$

$$\sum_{i=n}^{n+2k+1} U_{(2m+1)i+a}$$
$$= \frac{1}{V_{2m+1}}\begin{cases} U_{(2m+1)(k+1)}\left[V_{(2m+1)(n+k)+a} + V_{(2m+1)(n+k+1)+a}\right] & \text{if } k \text{ is odd,} \\ V_{(2m+1)(k+1)}\left[U_{(2m+1)(n+k)+a} + U_{(2m+1)(n+k+1)+a}\right] & \text{if } k \text{ is even,} \end{cases}$$

$$\sum_{i=n}^{n+2k+1} U_{n-2mi} = (-1)^{n-1}\begin{cases} U_{2m(k+1)}V_{2m(n+k)+m-n}/V_m & \text{if } m \text{ is odd,} \\ U_{2m(k+1)}U_{2m(n+k)+m-n}/U_m & \text{if } m \text{ is even,} \end{cases}$$

$$\sum_{i=n}^{n+2k+1} V_{i+a} = \begin{cases} \Delta U_{k+1}U_{n+k+a+2} & \text{if } k \text{ is odd,} \\ V_{k+1}V_{n+k+a+2} & \text{if } k \text{ is even,} \end{cases}$$

$$\sum_{i=n}^{n+2k+1} V_{2mi+a} = \begin{cases} \Delta U_{2m(k+1)}U_{2m(n+k+1)+a-m}/V_m & \text{if } m \text{ is odd,} \\ U_{2m(k+1)}V_{2m(n+k+1)+a-m}/U_m & \text{if } m \text{ is even,} \end{cases}$$

$$\sum_{i=n}^{n+2k+1} V_{(2m+1)i+a}$$
$$= \frac{1}{V_{2m+1}}\begin{cases} \Delta U_{(2m+1)(k+1)}\left[U_{(2m+1)(n+k)+a} + U_{(2m+1)(n+k+1)+a}\right] & \text{if } k \text{ is odd,} \\ V_{(2m+1)(k+1)}\left[V_{(2m+1)(n+k)+a} + V_{(2m+1)(n+k+1)+a}\right] & \text{if } k \text{ is even,} \end{cases}$$

$$\sum_{i=n}^{n+2k+1} V_{n-2mi} = (-1)^{n}\begin{cases} \Delta U_{2m(k+1)}U_{m(2n+2k+1)-n}/V_m & \text{if } m \text{ is odd,} \\ U_{2m(k+1)}V_{m(2n+2k+1)-n}/U_m & \text{if } m \text{ is even.} \end{cases}$$

Proof. We only prove the first identity. The others could be similarly proven. By the Binet formulas of $\{U_n\}$ and $\{V_n\}$, we write that for even k,

$$\begin{aligned} &\sum_{i=n}^{n+2k+1} U_{i+a} \\ &= \sum_{i=n}^{n+2k+1}\left(\frac{\alpha^{i+a}-\beta^{i+a}}{\Delta}\right) \\ &= \frac{1}{\Delta}\left(\alpha^{n+a}-\beta^{n+a}+\alpha^{n+1+a}-\beta^{n+1+a}+\ldots+\alpha^{n+2k+1+a}-\beta^{n+2k+1+a}\right) \\ &= \frac{1}{\Delta}\left[\alpha^{n+a}\left(1+\alpha+\ldots+\alpha^{2k+1}\right)-\beta^{n+a}\left(1+\beta+\ldots+\beta^{2k+1}\right)\right] \\ &= \frac{1}{\Delta}\left[\frac{\alpha^{n+a}-\alpha^{n+a+2k+2}}{1-\alpha}-\frac{\beta^{n+a}-\beta^{n+a+2k+2}}{1-\beta}\right] \end{aligned}$$

$$= \frac{1}{\Delta(-p)}(\alpha^{n+a} - \alpha^{n+a+2k+2} + \alpha^{n+a-1} - \alpha^{n+a+2k+1} - \beta^{n+a}$$
$$+ \beta^{n+a+2k+2} - \beta^{n+a-1} + \beta^{n+a+2k+1})$$
$$= \frac{1}{p\Delta}\left(\left(\alpha^{k+1} + \beta^{k+1}\right)\left(\alpha^{n+k+a+1} - \beta^{n+k+a+1}\right) - \left(\alpha^{n+a-1} - \beta^{n+a-1}\right)\right.$$
$$\left. + \left(\alpha^{n+2k+a+1} - \beta^{n+2k+a+1}\right)\right)$$
$$= \frac{1}{p}\left(V_{k+1}U_{n+k+a+1} - U_{n+a-1} + U_{n+2k+a+1}\right),$$

as claimed for even k. The proof is similarly obtained for odd k. □

Theorem 2.2. *For $k \geq 0$, and, any integers n, m, a and b,*

$$\sum_{i=n}^{n+2k+1} U_{(2m+1)i+a}U_{(2m+1)i+b} = \frac{U_{2(k+1)(2m+1)}U_{2(n+k+1)(2m+1)-2m+a+b-1}}{V_{2m+1}},$$

$$\sum_{i=n}^{n+2k+1} U_{(2m+1)i+a}V_{(2m+1)i+b} = \frac{U_{2(k+1)(2m+1)}V_{2(n+k+1)(2m+1)-2m+a+b-1}}{V_{2m+1}},$$

and

$$\sum_{i=n}^{n+2k+1} V_{(2m+1)i+a}V_{(2m+1)i+b} = \frac{U_{2(k+1)(2m+1)}}{V_{2m+1}}$$
$$\times\left[V_{2(2m+1)(n+k+1)-2m+a+b} + V_{2(2m+1)(n+k+1)-2(m+1)+a+b}\right].$$

Proof. We only prove the first identity for the case $m = 0$, that is,

$$\sum_{i=n}^{n+2k+1} U_{i+a}U_{i+b} = \frac{U_{2(k+1)}U_{2(n+k)+a+b+1}}{U_2}.$$

The others are similarly proven. Now consider

$$\sum_{i=n}^{n+2k+1} U_{i+a}U_{i+b}$$
$$= \frac{1}{\Delta^2}\sum_{i=n}^{n+2k+1}\left(\alpha^{i+a} - \beta^{i+a}\right)\left(\alpha^{i+b} - \beta^{i+b}\right)$$
$$= \frac{1}{\Delta^2}\sum_{i=n}^{n+2k+1}\left(\alpha^{2i+a+b} + \beta^{2i+a+b} - (-1)^{i+a}\left(\alpha^t + \beta^t\right)\right)$$
$$= \frac{1}{\Delta^2}\left(\alpha^{2n+a+b} - \beta^{2n+a+b} + \alpha^{2(n+1)+a+b} - \beta^{2(n+1)+a+b} + \ldots\right.$$
$$\left. + \alpha^{2(n+2k+1)+a+b} - \beta^{2(n+2k+1)+a+b}\right)$$
$$= \frac{1}{\Delta^2}\left(\alpha^{2n+a+b}\left(1 + \alpha^2 + \ldots + \alpha^{2(2k+1)}\right) + \beta^{2n+a+b}\left(1 + \beta^2 + \ldots + \beta^{2(2k+1)}\right)\right)$$
$$= \frac{1}{\Delta^2}\left(\frac{\alpha^{2n+a+b} - \alpha^{2n+4k+a+b+4}}{-p\alpha} + \frac{\beta^{2n+a+b} - \beta^{2n+4k+a+b+4}}{-p\beta}\right)$$
$$= -\frac{1}{p\Delta^2}\left(\alpha^{2n+a+b-1} - \alpha^{2n+4k+a+b+3} + \beta^{2n+a+b-1} - \beta^{2n+4k+a+b+3}\right)$$
$$= \left(\alpha^{2(k+1)} - \beta^{2(k+1)}\right)\left(\alpha^{2(n+k)+a+b+1} - \beta^{2(n+k)+a+b+1}\right)/p$$
$$= U_{2(k+1)}U_{2(n+k)+a+b+1}/U_2,$$

as claimed. □

As an example, for any integers a and b, we get

$$\sum_{i=0}^{t} U_{i+a}V_{i+b} = U_{t+1}V_{t+a+b}/2.$$

Theorem 2.3. *For $k \geq 0$, and, any integers n and m,*

$$\begin{aligned} \sum_{i=n}^{n+2k+1} U_{n-(2m+1)i}^2 &= U_{4m(n+k)+2(m+k)+1}U_{2(2m+1)(k+1)}/V_{2m+1}, \\ \sum_{i=n}^{n+2k+1} V_{n-(2m+1)i}^2 &= \Delta U_{4m(n+k)+2(m+k)+1}U_{2(2m+1)(k+1)}/V_{2m+1}. \end{aligned}$$

Proof. For the first identity, consider

$$\begin{aligned} &\sum_{i=n}^{n+2k+1} U_{n-(2m+1)i}^2 \\ = \ & \frac{1}{\Delta^2} \sum_{i=n}^{n+2k+1} \left(\alpha^{n-(2m+1)i} - \beta^{n-(2m+1)i}\right)^2 \\ = \ & \frac{1}{\Delta^2} \sum_{i=n}^{n+2k+1} \left(\alpha^{2n-2(2m+1)i} + \beta^{2n-2(2m+1)i} - 2(-1)^{n-(2m+1)i}\right) \\ = \ & \frac{1}{\Delta^2}(\alpha^{2n-2(2m+1)n} + \beta^{2n-2(2m+1)n} + \alpha^{2n-2(2m+1)(n+1)} \\ & +\beta^{2n-2(2m+1)(n+1)} + ... + \alpha^{2n-2(2m+1)(n+2k+1)} + \beta^{2n-2(2m+1)(n+2k+1)}\Big) \\ = \ & \frac{1}{\Delta^2}\left(\alpha^{2n-2(2m+1)n}\left(1 + \alpha^{-2(2m+1)} + ... + \alpha^{-2(2m+1)(2k+1)}\right)\right. \\ & \left. +\beta^{2n-2(2m+1)n}\left(1 + \beta^{-2(2m+1)} + ... + \beta^{-2(2m+1)(2k+1)}\right)\right) \\ = \ & \frac{1}{\Delta^2}\left(\frac{\alpha^{-4nm}\left(1-\alpha^{-2(2m+1)(2k+2)}\right)}{1-\alpha^{-2(2m+1)}} + \frac{\beta^{-4nm}\left(1-\beta^{-2(2m+1)(2k+2)}\right)}{1-\beta^{-2(2m+1)}}\right) \\ = \ & \frac{-\left(\alpha^{2m+1}+\beta^{2m+1}\right)}{\Delta^2\left(\alpha^{2m+1}+\beta^{2m+1}\right)^2}\left(\alpha^{4m(n+k)+2(m+k)+1} - \beta^{4m(n+k)+2(m+k)+1}\right) \\ & \times\left(\alpha^{2(2m+1)(k+1)} - \beta^{2(2m+1)(k+1)}\right) \\ = \ & U_{4m(n+k)+2(m+k)+1}U_{2(2m+1)(k+1)}/V_{2m+1}, \end{aligned}$$

as claimed. The second identity could be proven similar. □

As an interesting example, we note that

$$\sum_{i=0}^{n} U_{mi}^2 = \frac{U_{nm}^2 V_{mn}}{V_m}.$$

3 Alternating sums for generalized Fibonacci and Lucas numbers

In this section, we present certain alternating sums including generalized Fibonacci and Lucas numbers.

Theorem 3.1. *For $k \geq 0$, and, any integers n and a,*

$$\sum_{i=n}^{n+2k+1} (-1)^i U_{i+a} = \frac{(-1)^n}{U_2}\begin{cases} U_{k+1}V_{n+k+a} + U_{n+a} - U_{n+2k+a+2} & \text{if } k \text{ is odd}, \\ V_{k+1}U_{n+k+a} + U_{n+a} - U_{n+2k+a+2} & \text{if } k \text{ is even}, \end{cases}$$

$$\sum_{i=n}^{n+2k+1} (-1)^i U_{2mi+a} = (-1)^{n-1}\begin{cases} U_{2m(k+1)}U_{2m(n+k+1)+a-m}/U_m & \text{if } m \text{ is odd}, \\ U_{2m(k+1)}V_{2m(n+k+1)+a-m}/V_m & \text{if } m \text{ is even}, \end{cases}$$

$$\sum_{i=n}^{n+2k+1}(-1)^i U_{(2m+1)i+a}=\frac{(-1)^n}{V_{2m+1}}$$
$$\times\begin{cases} U_{(2m+1)(k+1)}\left(V_{(2m+1)(n+k)+a}-V_{(2m+1)(n+k+1)+a}\right) & \text{if } k \text{ is odd,}\\ V_{(2m+1)(k+1)}\left(U_{(2m+1)(n+k)+a}-U_{(2m+1)(n+k+1)+a}\right) & \text{if } k \text{ is even,}\end{cases}$$

$$\sum_{i=n}^{n+2k+1}(-1)^i U_{n-2mi}=\begin{cases} U_{2m(k+1)}U_{2m(n+k)+m-n}/U_m & \text{if } m \text{ is odd,}\\ U_{2m(k+1)}V_{2m(n+k)+m-n}/V_m & \text{if } m \text{ is even,}\end{cases}$$

$$\sum_{i=n}^{n+2k+1}(-1)^i V_{i+a}=(-1)^{n-1}\begin{cases} \Delta U_{k+1}U_{n+k+a-1} & \text{if } k \text{ is odd,}\\ V_{k+1}V_{n+k+a-1} & \text{if } k \text{ is even,}\end{cases}$$

$$\sum_{i=n}^{n+2k+1}(-1)^i V_{2mi+a}=(-1)^{n-1}\begin{cases} U_{2m(k+1)}V_{2m(n+k+1)+a-m}/U_m & \text{if } m \text{ is odd,}\\ \Delta U_{2m(k+1)}U_{2m(n+k+1)+a-m}/V_m & \text{if } m \text{ is even,}\end{cases}$$

$$\sum_{i=n}^{n+2k+1}(-1)^i V_{(2m+1)i+a}=\frac{(-1)^{n-1}}{V_{2m+1}}$$
$$\times\begin{cases} \Delta U_{(2m+1)(k+1)}\left(U_{(2m+1)(n+k+1)+a}-U_{(2m+1)(n+k)+a}\right) & \text{if } k \text{ is odd,}\\ V_{(2m+1)(k+1)}\left(V_{(2m+1)(n+k+1)+a}-V_{(2m+1)(n+k)+a}\right) & \text{if } k \text{ is even,}\end{cases}$$

$$\sum_{i=n}^{n+2k+1}(-1)^i V_{n-2mi}=\begin{cases} -U_{2m(k+1)}V_{2m(n+k)+m-n}/U_m & \text{if } m \text{ is odd,}\\ -\Delta U_{2m(k+1)}U_{2m(n+k)+m-n}/V_m & \text{if } m \text{ is even.}\end{cases} .$$

Proof. For odd k, we write

$$\sum_{i=n}^{n+2k+1}(-1)^i U_{i+a}=\frac{1}{\Delta}\sum_{i=n}^{n+2k+1}(-1)^i\left(\alpha^{i+a}-\beta^{i+a}\right)$$
$$=\frac{(-1)^n}{\Delta}\left(\alpha^{n+a}-\beta^{n+a}-\alpha^{n+1+a}+\beta^{n+1+a}+...-\alpha^{n+2k+1+a}+\beta^{n+2k+1+a}\right)$$
$$=\frac{(-1)^n}{\Delta}(\alpha^{n+a}\left(1+(-\alpha)+(-\alpha)^3+...+(-\alpha)^{2k+1}\right)$$
$$-\beta^{n+a}\left(1+(-\beta)+(-\beta)^3+...+(-\beta)^{2k+1}\right))$$
$$=\frac{(-1)^n}{\Delta}\left(\frac{\alpha^{n+a}-\alpha^{n+a+2k+2}}{1+\alpha}-\frac{\beta^{n+a}-\beta^{n+a+2k+2}}{1+\beta}\right)$$
$$=\frac{(-1)^n}{\Delta p}(\alpha^{n+a}-\alpha^{n+a+2k+2}-\alpha^{n+a-1}+\alpha^{n+a+2k+1}$$
$$-\beta^{n+a}+\beta^{n+a+2k+2}-\beta^{n+a-1}-\beta^{n+a+2k+1})$$
$$=\frac{(-1)^n}{p\Delta}((\alpha^{k+1}-\beta^{k+1})(\alpha^{n+k+a}+\beta^{n+k+a})$$
$$+(\alpha^{n+a}-\beta^{n+a})-(\alpha^{n+2k+a+2}-\beta^{n+2k+a+2}))$$
$$=\frac{(-1)^n}{p}\left(U_{k+1}V_{n+k+a}+U_{n+a}-U_{n+2k+a+2}\right),$$

which completes the proof for odd k. The proof is similarly obtained for even k. The others could be similarly proven. □

Theorem 3.2. *For $k\geq 0$, and, any integers n, a and b,*

$$\sum_{i=n}^{n+2k+1}(-1)^i U_{2mi+a}U_{2mi+b}=\frac{(-1)^{n-1}}{V_{2m}}U_{4m(k+1)}U_{4m(n+k)+2m+a+b},$$

$$\sum_{i=n}^{n+2k+1}(-1)^i U_{2mi+a}V_{2mi+b}=\frac{(-1)^{n-1}}{V_{2m}}U_{4m(k+1)}V_{4m(n+k)+2m+a+b},$$

$$\sum_{i=n}^{n+2k+1}(-1)^i V_{2mi+a}V_{2mi+b} = (-1)^{n-1}\frac{U_{4m(k+1)}}{V_{2m}}\left(V_{4m(n+k)+2m+a+b+1}+V_{4m(n+k)+2m+a+b-1}\right).$$

Proof. We prove the first identity. The others could be similarly proven. By the Binet formula of $\{U_n\}$, we write

$$\begin{aligned}
&\sum_{i=n}^{n+2k+1}(-1)^i U_{2mi+a}U_{2mi+b}\\
&=\frac{1}{\Delta^2}\sum_{i=n}^{n+2k+1}(-1)^i\left(\alpha^{2mi+a}-\beta^{2mi+a}\right)\left(\alpha^{2mi+b}-\beta^{2mi+b}\right)\\
&=\frac{1}{\Delta^2}\sum_{i=n}^{n+2k+1}(-1)^i\left(\alpha^{4mi+a+b}+\beta^{4mi+a+b}-(-1)^a\left(\alpha^t+\beta^t\right)\right)\\
&=\frac{(-1)^n}{\Delta^2}\Big(\left(\alpha^{4mn+a+b}+\beta^{4mn+a+b}\right)-\left(\alpha^{4m(n+1)+a+b}+\beta^{4m(n+1)+a+b}\right)+\ldots\\
&-\left(\alpha^{4m(n+2k+1)+a+b}+\beta^{4m(n+2k+1)+a+b}\right)\Big)\\
&=\frac{(-1)^n}{\Delta^2}\Big(\alpha^{4mn+a+b}\left(1+\left(-\alpha^{4m}\right)+\ldots+\left(-\alpha^{4m}\right)^{2k+1}\right)\\
&+\beta^{4mn+a+b}\left(1+\left(-\beta^{4m}\right)+\ldots+\left(-\beta^{4m}\right)^{2k+1}\right)\Big)\\
&=\frac{(-1)^n}{\Delta^2}\left(\frac{\alpha^{4mn+a+b}-\alpha^{4m(n+2k+2)+a+b}}{1+\alpha^{4m}}+\frac{\beta^{4mn+a+b}-\beta^{4m(n+2k+2)+a+b}}{1+\beta^{4m}}\right)\\
&=\frac{(-1)^{n-1}}{\Delta^2\left(\alpha^{2m}+\beta^{2m}\right)^2}\left(\alpha^{2m}+\beta^{2m}\right)\left(\alpha^{4m(k+1)}-\beta^{4m(k+1)}\right)\\
&\times\left(\alpha^{4m(n+k)+2m+a+b}-\beta^{4m(n+k)+2m+a+b}\right)\\
&=(-1)^{n-1}U_{4m(k+1)}U_{4m(n+k)+2m+a+b}/V_{2m},
\end{aligned}$$

as claimed. □

Theorem 3.3. *For $k \geq 0$, and, any integers n and m*

$$\begin{aligned}
\sum_{i=n}^{n+2k+1}(-1)^i U_{n-2mi}^2 &= (-1)^{n-1}U_{4m(n+k)+2(m-n)}U_{4m(k+1)}/V_{2m},\\
\sum_{i=n}^{n+2k+1}(-1)^i V_{n-2mi}^2 &= (-1)^{n-1}\Delta U_{4m(n+k)+2(m-n)}U_{4m(k+1)}/V_{2m}.
\end{aligned}$$

Proof. Consider

$$\begin{aligned}
&\sum_{i=n}^{n+2k+1}(-1)^i U_{n-2mi}^2\\
=&\ \frac{1}{\Delta^2}\sum_{i=n}^{n+2k+1}(-1)^i\left(\alpha^{n-2mi}-\beta^{n-2mi}\right)^2\\
=&\ \frac{1}{\Delta^2}\sum_{i=n}^{n+2k+1}(-1)^i\left(\alpha^{2n-4mi}+\beta^{2n-4mi}-2(-1)^n\right)\\
=&\ \frac{(-1)^n}{\Delta^2}(\left(\alpha^{2n-4mn}+\beta^{2n-4mn}\right)-\left(\alpha^{2n-4m(n+1)}+\beta^{2n-4m(n+1)}\right)+\ldots\\
&-\left(\alpha^{2n-4m(n+2k+1)}+\beta^{2n-4m(n+2k+1)}\right))
\end{aligned}$$

$$\begin{aligned}
&= \frac{(-1)^n}{\Delta^2}(\alpha^{2n-4mn}\left(1+\left(-\alpha^{-4m}\right)+...+\left(-\alpha^{-4m}\right)^{2k+1}\right)\\
&\quad +\beta^{2n-4mn}\left(1+\left(-\beta^{-4m}\right)+...+\left(-\beta^{-4m}\right)^{2k+1}\right))\\
&= \frac{(-1)^n}{\Delta^2}\left(\alpha^{2n-4mn}\frac{1-\alpha^{-4m(2k+2)}}{1+\alpha^{-4m}}+\beta^{2n-4mn}\frac{1-\beta^{-4m(2k+2)}}{1+\beta^{-4m}}\right)\\
&= \frac{(-1)^n}{\Delta^2 V_{2m}^2}(\alpha^{2n-4mn}-\alpha^{4m(2k-n+2)+2n}+\alpha^{2n-4mn-4m}-\alpha^{4m(2k-n+1)+2n}\\
&\quad +\beta^{2n-4mn}-\beta^{4m(2k-n+2)+2n}+\beta^{2n-4mn-4m}-\beta^{4m(2k-n+1)+2n})\\
&= \frac{1}{\Delta^2 V_{2m}^2}(-1)^{n-1}\left(\alpha^{2m}+\beta^{2m}\right)\left(\alpha^{4m(k+1)}-\beta^{4m(k+1)}\right)\times\\
&\quad \left(\alpha^{4m(n+k)+2(m-n)}-\beta^{4m(n+k)+2(m-n)}\right)\\
&= (-1)^{n-1}U_{4m(n+k)+2(m-n)}U_{4m(k+1)}/V_{2m},
\end{aligned}$$

as claimed. The second identity for the generalized Lucas number counterpart is similarly obtained. □

For example, we get

$$\sum_{i=1}^{2t}(-1)^i U_{2^{r+1}i}=\frac{1}{V_{2^r}}U_{2^{r+1}t}V_{(2t+1)2^r}.$$

Theorem 3.4. *For $k\geq 0$, and, any integers n and m,*

$$\sum_{i=n}^{n+2k+1}(-1)^i U_iU_{mn-i}=-\left(U_{n(m-2)-(2k+1)}U_{2(k+1)}\right)/U_2,$$

$$\sum_{i=n}^{n+2k+1}(-1)^i U_{mi}U_{n-mi}=\begin{cases}(-1)^{n-1}U_{2m(n+k)+m-n}U_{2m(k+1)}/V_m & \text{if } m \text{ is odd,}\\ U_{2m(n+k)+m-n}U_{2m(k+1)}/V_m & \text{if } m \text{ is even,}\end{cases}$$

$$\sum_{i=n}^{n+2k+1}(-1)^i V_iV_{mn-i}=\Delta U_{n(m-2)-(2k+1)}U_{2(k+1)}/U_2,$$

$$\sum_{i=n}^{n+2k+1}(-1)^i V_{mi}V_{n-mi}=-\Delta U_{2m(n+k)+m-n}U_{2m(k+1)}/V_m,$$

$$\sum_{i=n}^{n+2k+1}(-1)^i U_iV_{mn-i}=-V_{n(m-2)-(2k+1)}U_{2(k+1)}/U_2,$$

$$\sum_{i=n}^{n+2k+1}(-1)^i V_iU_{mn-i}=V_{n(m-2)-(2k+1)}U_{2(k+1)}/U_2,$$

$$\sum_{i=n}^{n+2k+1}(-1)^i U_{mi}V_{n-mi}=\begin{cases}(-1)^{n}V_{2m(n+k)+m-n}U_{2m(k+1)}/V_m & \text{if } m \text{ is odd,}\\ -V_{2m(n+k)+m-n}U_{2m(k+1)}/V_m & \text{if } m \text{ is even,}\end{cases}$$

and

$$\sum_{i=n}^{n+2k+1}(-1)^i V_{mi}U_{n-mi}=\begin{cases}(-1)^{n-1}V_{2m(n+k)+m-n}U_{2m(k+1)}/V_m & \text{if } m \text{ is odd,}\\ V_{2m(n+k)+m-n}U_{2m(k+1)}/V_m & \text{if } m \text{ is even.}\end{cases}$$

Proof. Consider

$$\begin{aligned}
&\sum_{i=n}^{n+2k+1} (-1)^i U_i U_{mn-i} \\
&= \frac{1}{\Delta^2} \sum_{i=n}^{n+2k+1} (-1)^i \left(\alpha^i - \beta^i\right)\left(\alpha^{mn-i} - \beta^{mn-i}\right) \\
&= \frac{1}{\Delta^2} \sum_{i=n}^{n+2k+1} (-1)^i \left(\alpha^{mn} + \beta^{mn} - \alpha^{mn-i}\beta^i - \beta^{mn-i}\alpha^i\right) \\
&= \frac{1}{\Delta^2} \sum_{i=n}^{n+2k+1} (-1)^i \left(-(-1)^i \alpha^{mn-2i} - (-1)^i \beta^{mn-2i}\right) \\
&= \frac{-1}{\Delta^2} \sum_{i=n}^{n+2k+1} \left(\alpha^{mn-2i} + \beta^{mn-2i}\right) \\
&= \frac{-1}{\Delta^2}(\left(\alpha^{mn-2n} + \beta^{mn-2n}\right) + \left(\alpha^{mn-2(n+1)} + \beta^{mn-2(n+1)}\right) \\
&+ \ldots + \left(\alpha^{mn-2(n+2k+1)} + \beta^{mn-2(n+2k+1)}\right)) \\
&= \frac{-1}{\Delta^2}(\alpha^{mn-2n}\left(1 + \alpha^{-2} + \ldots + \alpha^{-2(2k+1)}\right) \\
&+ \beta^{mn-2n}\left(1 + \beta^{-2} + \ldots + \beta^{-2(2k+1)}\right)) \\
&= \frac{-1}{\Delta^2}\left[\alpha^{mn-2n}\frac{1-\alpha^{-2(2k+2)}}{1-\alpha^{-2}} + \beta^{mn-2n}\frac{1-\beta^{-2(2k+2)}}{1-\beta^{-2}}\right] \\
&= \frac{-1}{\Delta^2(\alpha+\beta)^2}(\alpha+\beta)\left(\alpha^{n(m-2)-(2k+1)} - \beta^{n(m-2)-(2k+1)}\right)\left(\alpha^{2(k+1)} - \beta^{2(k+1)}\right) \\
&= -U_{n(m-2)-(2k+1)}U_{2(k+1)}/U_2,
\end{aligned}$$

as claimed. The others could be similarly proven. □

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