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Decompositions of the Cauchy and Ferrers-Jackson Polynomials

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Abstract. Recently Witula and Slota give decompositions of the Cauchy and Ferrers-Jackson polynomials [Cauchy, Ferrers-Jackson and Chebyshev polynomials and identities for the powers of elements of some conjugate recurrence sequences, Central Europan J. Math., 2006]. Our main purpose is to derive different decomposition of the Cauchy and Ferrers-Jackson polynomials. Our approach is to use the Waring formula and Saalschütz's identity to prove claimed results. Also we obtain generalizations of the results of Carlitz, Hunter and Koshy as corollaries of our results about sums and differences of powers of the Fibonacci and Lucas numbers.
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1. Introduction

For $n \in \mathbb{N}$, the Cauchy and Ferrers-Jackson polynomials are defined by

$$p_n(x,y) := (x+y)^{2n+1} - x^{2n+1} - y^{2n+1}$$

and

$$q_n(x,y) := (x+y)^{2n} + x^{2n} + y^{2n},$$

respectively. Some authors have studied on their decompositions. Recently Witula and Slota [8] obtained the following decompositions

$$p_n(x,y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{2n+1}{n-k} \binom{n-k}{2k+1} (xy(x+y))^{2k+1} \left(x^2+xy+y^2\right)^{n-3k-1} \tag{1}$$

and

$$q_n(x,y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{2n}{n-k} \binom{n-k}{2k} (xy(x+y))^{2k} \left(x^2+xy+y^2\right)^{n-3k}. \tag{2}$$

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The proof for these decompositions were given by induction that are based on simple recurrence dependence between polynomials $p_n(x,y)$ and $q_n(x,y)$.

In this paper our main purpose is to derive alternative approach to obtain different decompositions of the polynomials Cauchy and Ferrers-Jackson. To prove the claimed result, our approach is to use the Waring formula and Saalschütz's identity given by

$$a^m + b^m = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \frac{m}{m-k} \binom{m-k}{k} (ab)^k (a+b)^{m-2k}, \quad m > 0,$$

and for $n \geq 0$

$$F\left(\left. \begin{matrix} a, b, -n \\ c, a+b-c-n+1 \end{matrix} \right| 1 \right) = \frac{(c-a)^{\overline{n}} (c-b)^{\overline{n}}}{c^{\overline{n}} (c-a-b)^{\overline{n}}},$$

respectively, for more details see [4].

It would be much valuable to note that the decompositions of the Cauchy and Ferrers-Jackson polynomials are very closely related to the identities given by Carlitz, Hunter and Koshy on differences and sums of powers of Fibonacci and Lucas numbers up to seventh and eighth powers, respectively.

We also present generalizations of the identities Carlitz [1], Hunter and Koshy [5] including Fibonacci and Lucas numbers for any odd and even powers as applications of our main results

Recall that the well-known Fibonacci sequence $\{F_n\}$ is defined by the recurrence

$$F_n = F_{n-1} + F_{n-2}, \quad n > 1$$

with initials $F_0 = 0$ and $F_1 = 1$. The Lucas sequence $\{L_n\}$ satisfies the same recurrence relation but the initials are $L_0 = 2$ and $L_1 = 1$. A formula for the generating functions of powers of the general cases of these sequences and squaring terms of ℓ-sequence were also given in [6] and [7], respectively.

Now we recall the results of Carlitz [1]. He proposed following interesting identities as advanced problems for $n > 1$

$$\begin{aligned}
F_{n+1}^3 - F_n^3 - F_{n-1}^3 &= 3F_{n+1}F_nF_{n-1}, \\
L_{n+1}^3 - L_n^3 - L_{n-1}^3 &= 3L_{n+1}L_nL_{n-1}, \\
F_{n+1}^5 - F_n^5 - F_{n-1}^5 &= 5F_{n+1}F_nF_{n-1}\left(2F_n^2 + (-1)^n\right), \\
L_{n+1}^5 - L_n^5 - L_{n-1}^5 &= 5L_{n+1}L_nL_{n-1}\left(2L_n^2 - 5(-1)^n\right), \\
F_{n+1}^7 - F_n^7 - F_{n-1}^7 &= 7F_{n+1}F_nF_{n-1}\left(2F_n^2 + (-1)^n\right)^2, \\
L_{n+1}^7 - L_n^7 - L_{n-1}^7 &= 7L_{n+1}L_nL_{n-1}\left(2L_n^2 - 5(-1)^n\right)^2.
\end{aligned} \tag{3}$$

Then Carlitz proved his propositions two years later after he asked. Also Charles Wall solved the problem at the same time independently (see [2]). For the sums of the powers of Fibonacci and Lucas numbers, Carlitz, Hunter and Koshy gave the following identities,

$$\begin{aligned}
F_{n+1}^4 + F_n^4 + F_{n-1}^4 &= 2\left(2F_n^2 + (-1)^n\right)^2 \\
L_{n+1}^4 + L_n^4 + L_{n-1}^4 &= 2\left(2L_n^2 - 5\,(-1)^n\right)^2 \\
F_{n+1}^6 + F_n^6 + F_{n-1}^6 &= 2\left(2F_n^2 + (-1)^n\right)^3 + 3F_{n-1}^2 F_n^2 F_{n+1}^2 \\
F_{n+1}^8 + F_n^8 + F_{n-1}^8 &= 2\left(2F_n^2 + (-1)^n\right)^4 + 8F_{n-1}^2 F_n^2 \\
&\quad \times \left(F_{n-1}^4 + F_n^4 + 4F_{n-1}^2 F_n^2 + 3F_{n-1}F_nF_{2n-1}\right). \qquad (4)
\end{aligned}$$

2. Decomposition of the Cauchy and Ferrers-Jackson Polynomials

Before our main results, we give a lemma playing crucial point.

Lemma 1. *For $m > 0$ and $r \in \{0,1\}$, then*

$$\sum_{k=0}^{\lfloor \frac{t-r+1}{3} \rfloor} \binom{t-k+1}{t-3k-r+1}_{3,2} \binom{t-3k-r+1}{m-2k} = \frac{2t+r+2}{2t-m+2}\binom{2t-m+2}{m+r},$$

where

$$\binom{n}{k}_{3,2} = \frac{3n-k}{n}\binom{n}{k}.$$

Proof. Assume that $r = 1$. In much clear form we have to prove that

$$\sum_{k=0}^{\lfloor t/3 \rfloor} \binom{t+1-k}{t-3k}\binom{t-3k}{m-2k}\frac{1}{t+1-k} = \frac{1}{m+1}\binom{2t-m+1}{m}.$$

Consider the LHS of the claimed identity

$$\sum_{k=0}^{\lfloor t/3 \rfloor} \binom{t+1-k}{t-3k}\binom{t-3k}{m-2k}\frac{1}{t+1-k} = \sum_{k=0}^{\lfloor t/3 \rfloor} \frac{(t-k)!}{(2k+1)!\,(m-2k)!\,(t-m-k)!}.$$

Define

$$T_k := \frac{(t-k)!}{(2k+1)!\,(m-2k)!\,(t-m-k)!}.$$

Consider

$$\begin{aligned}
\frac{T_{k+1}}{T_k} &= \frac{(t-k-1)!\,(2k+1)!\,(m-2k)!\,(t-m-k)!}{(2k+3)!\,(m-2k-2)!\,(t-m-k-1)!\,(t-k)!} \\
&= \frac{(m-2k)\,(m-2k-1)\,(t-m-k)}{(2k+3)\,(2k+2)\,(t-k)} \\
&= \frac{\left(k-\frac{m}{2}\right)\left(k-\frac{m-1}{2}\right)(k+m-t)}{(k-t)\left(k+\frac{3}{2}\right)(k+1)}.
\end{aligned}$$

Since

$$\sum_{k=0}^{\lfloor t/3\rfloor}\binom{t+1-k}{t-3k}\binom{t-3k}{m-2k}\frac{1}{t+1-k}=T_0F\left(\begin{matrix}-\frac{m}{2}, & -\frac{m-1}{2}, & m-t\\ \frac{3}{2}, & -t\end{matrix}\middle|1\right),$$

the LHS of the sum just above yields Saalschütz's identity. Thus

$$\begin{aligned}\sum_{k=0}^{\lfloor t/3\rfloor}\binom{t+1-k}{t-3k}\binom{t-3k}{m-2k}\frac{1}{t+1-k}&=\binom{t}{m}F\left(\begin{matrix}-\frac{m}{2}, & -\frac{m-1}{2}, & m-t\\ \frac{3}{2}, & -t\end{matrix}\middle|1\right)\\&=\binom{t}{m}\frac{\left(\frac{m+3}{2}\right)^{\overline{t-m}}\left(\frac{m+2}{2}\right)^{\overline{t-m}}}{\left(\frac{3}{2}\right)^{\overline{t-m}}(m+1)^{\overline{t-m}}}\\&=\frac{1}{m+1}\binom{2t-m+1}{m},\end{aligned}$$

as claimed.

Now suppose that $r=0$. Thus claim takes the form

$$\sum_{k=0}^{\lfloor\frac{t+1}{3}\rfloor}\binom{t-k+1}{t-3k+1}\binom{t-3k+1}{m-2k}\frac{1}{t-k+1}=\frac{1}{2t-m+2}\binom{2t-m+2}{m}.$$

Define

$$T_k:=\binom{t-k+1}{t-3k+1}\binom{t-3k+1}{m-2k}\frac{1}{t-k+1}.$$

Thus

$$\begin{aligned}T_k&=\frac{(t-k+1)!}{(t-3k+1)!\,(2k)!}\frac{(t-3k+1)!}{(m-2k)!\,(t-m-k+1)!}\frac{1}{t-k+1}\\&=\frac{(t-k)!}{(2k)!\,(m-2k)!\,(t-m-k+1)!}\end{aligned}$$

and so

$$\begin{aligned}\frac{T_{k+1}}{T_k}&=\frac{(t-k-1)!}{(2k+2)!\,(m-2k-2)!\,(t-m-k)!}\frac{(2k)!\,(m-2k)!\,(t-m-k+1)!}{(t-k)!}\\&=\frac{(m-2k)\,(m-2k-1)\,(t-m-k+1)}{(t-k)\,(2k+2)\,(2k+1)}\\&=\frac{\left(k-\frac{m}{2}\right)\left(k-\frac{m-1}{2}\right)(k+m-t-1)}{(k-t)\,(k+1)\left(k+\frac{1}{2}\right)}\end{aligned}$$

Thus we write

$$\sum_{k=0}^{\lfloor\frac{t+1}{3}\rfloor}\binom{t-k+1}{t-3k+1}\binom{t-3k+1}{m-2k}\frac{1}{t-k+1}=T_0F\left(\begin{matrix}-\frac{m}{2}, & -\frac{m-1}{2}, & m-t-1\\ \frac{1}{2}, & -t\end{matrix}\middle|1\right),$$

which, by Saalschütz's identity and $T_0 = \binom{t+1}{m}\frac{1}{t+1}$, equals

$$\binom{t+1}{m}\frac{1}{t+1}\frac{\left(\frac{m+1}{2}\right)^{\overline{t-m+1}}\left(\frac{m}{2}\right)^{\overline{t-m+1}}}{\left(\frac{1}{2}\right)^{\overline{t-m+1}}m^{\overline{t-m+1}}} = \frac{1}{2t-m+2}\binom{2t-m+2}{m},$$

as claimed. $\square$

Now we are ready to give our main first result:

Theorem 1. *For $m \geq 0$ and $r \in \{0,1\}$, then*

$$\begin{aligned}&(a+b)^{2m+r} + (-a)^{2m+r} + (-b)^{2m+r}\\ &= \sum_{i=0}^{\lfloor (m-r)/3\rfloor}\binom{m-i}{m-3i-r}_{3,2}(ab(a+b))^{2i+r}\left(a^2+ab+b^2\right)^{m-3i-r}.\end{aligned} \tag{5}$$

Proof. Firstly, assume that $r = 0$. By the Waring formula, we have

$$\begin{aligned}&(a+b)^{2m} + a^{2m} + b^{2m}\\ &= (a+b)^{2m} + \sum_{k=0}^{m}\frac{2m}{2m-k}\binom{2m-k}{k}(-1)^k(ab)^k(a+b)^{2m-2k}.\end{aligned}$$

Then

$$\begin{aligned}&(a+b)^{2m} + \sum_{k=0}^{m}\frac{2m}{2m-k}\binom{2m-k}{k}(-1)^k(ab)^k(a+b)^{2m-2k}\\ &= 2(a+b)^{2m} + \sum_{k=1}^{m}\frac{2m}{2m-k}\binom{2m-k}{k}(-1)^k(ab)^k(a+b)^{2m-2k}\\ &= \binom{m}{m}_{3,2}(a+b)^{2m} + \sum_{k=1}^{m}\frac{2m}{2m-k}\binom{2m-k}{k}(-1)^k(ab)^k(a+b)^{2m-2k}.\end{aligned}$$

which, by the Waring formula, yields

$$\begin{aligned}&\binom{m}{m}_{3,2}(a+b)^{2m} + \sum_{k=1}^{m}\frac{2m}{2m-k}\binom{2m-k}{k}(-1)^k(ab)^k(a+b)^{2m-2k}\\ &= \sum_{k=0}^{m}\sum_{i=0}^{\lfloor\frac{m}{3}\rfloor}\binom{m-i}{m-3i}_{3,2}\binom{m-3i}{k-2i}(-1)^k(ab)^k(a+b)^{2m-2k}\\ &= \sum_{i=0}^{\lfloor\frac{m}{3}\rfloor}\binom{m-i}{m-3i}_{3,2}\sum_{k=2i}^{m-i}\binom{m-3i}{k-2i}(-1)^k(ab)^k(a+b)^{2m-2k}\\ &= \sum_{i=0}^{\lfloor\frac{m}{3}\rfloor}\binom{m-i}{m-3i}_{3,2}\sum_{k=0}^{m-3i}\binom{m-3i}{k}(-1)^k(ab)^{k+2i}(a+b)^{2m-2k-4i}\\ &= \sum_{i=0}^{\lfloor\frac{m}{3}\rfloor}\binom{m-i}{m-3i}_{3,2}(ab+(a+b))^{2i}\left((a+b)^2-ab\right)^{m-3i}\end{aligned}$$

which completes the proof for the case $r = 0$.

Consider for the case $r = 1$.

$$\begin{aligned}
&(a+b)^{2m+1} - a^{2m+1} - b^{2m+1} \\
&= (a+b)^{2m+1} - \sum_{k=0}^{m} (-1)^k \frac{2m+1}{2m+1-k} \binom{2m+1-k}{k} (ab)^k (a+b)^{2m+1-2k} \\
&= \sum_{k=1}^{m} (-1)^{k+1} \frac{2m+1}{2m+1-k} \binom{2m+1-k}{k} (ab)^k (a+b)^{2m+1-2k} \\
&= \sum_{k=0}^{m-1} (-1)^k \frac{2m+1}{2m-k} \binom{2m-k}{k+1} (ab)^{k+1} (a+b)^{2m-2k-1},
\end{aligned}$$

which, by the Waring formula, equals

$$\begin{aligned}
&\sum_{k=0}^{m-1} (-1)^k \sum_{i=0}^{\lfloor (m-1)/3 \rfloor} \binom{m-i}{m-3i-1}_{3,2} \binom{m-3i-1}{k-2i} (ab)^{k+1} (a+b)^{2m-2k-1} \\
&= \sum_{i=0}^{\lfloor (m-1)/3 \rfloor} \binom{m-i}{m-3i-1}_{3,2} \sum_{k=0}^{m-1} \binom{m-3i-1}{k-2i} (-1)^k (ab)^{k+1} (a+b)^{2m-2k-1} \\
&= \sum_{i=0}^{\lfloor (m-1)/3 \rfloor} \binom{m-i}{m-3i-1}_{3,2} (ab(a+b))^{2i+1} \left((a+b)^2 - ab \right)^{m-3i-1} \\
&= \sum_{i=0}^{\lfloor (m-1)/3 \rfloor} \binom{m-i}{m-3i-1}_{3,2} (ab(a+b))^{2i+1} \left((a+b)^2 - ab \right)^{m-3i-1}.
\end{aligned}$$

□

2.1. Generalizations of the results of Carlitz

In this section, we give general form of the identities (3) and (4) as corollaries of Theorem 1. If we take $a = F_n$ and $b = F_{n-1}$ in (5) with the case $r = 1$, then we get

$$\begin{aligned}
&F_{n+1}^{2m+1} - F_n^{2m+1} - F_{n-1}^{2m+1} \\
&= \sum_{i=0}^{\lfloor (m-1)/3 \rfloor} \binom{m-i}{m-3i-1}_{3,2} (F_{n+1}F_nF_{n-1})^{2i+1} \left(F_{n+1}^2 - F_nF_{n-1} \right)^{m-1-3i} \\
&= \sum_{i=0}^{\lfloor (m-1)/3 \rfloor} \binom{m-i}{m-3i-1}_{3,2} (F_{n+1}F_nF_{n-1})^{2i+1} \left(2F_n^2 + (-1)^n \right)^{m-1-3i}. \qquad (6)
\end{aligned}$$

Similarly, when $a = L_n$ and $b = L_{n-1}$, we obtain

$$\begin{aligned}
&L_{n+1}^{2m+1} - L_n^{2m+1} - L_{n-1}^{2m+1} \\
&= \sum_{i=0}^{\lfloor (m-1)/3 \rfloor} \binom{m-i}{m-3i-1}_{3,2} \left(L_{n+1}L_nL_{n-1}\right)^{2i+1} \left(L_{n+1}^2 - L_nL_{n-1}\right)^{m-1-3i} \\
&= \sum_{i=0}^{\lfloor (m-1)/3 \rfloor} \binom{m-i}{m-3i-1}_{3,2} \left(L_{n+1}L_nL_{n-1}\right)^{2i+1} \left(2L_n^2 - 5\left(-1\right)^n\right)^{m-1-3i}, \qquad (7)
\end{aligned}$$

For $r = 0$, we have the following identities

$$\begin{aligned}
&F_{n+1}^{2m} + F_n^{2m} + F_{n-1}^{2m} \\
&= \sum_{i=0}^{\lfloor m/3 \rfloor} \binom{m-i}{m-3i}_{3,2} \left(F_{n+1}F_nF_{n-1}\right)^{2i} \left(F_{n+1}^2 - F_nF_{n-1}\right)^{m-3i} \\
&= \sum_{i=0}^{\lfloor m/3 \rfloor} \binom{m-i}{m-3i}_{3,2} \left(F_{n+1}F_nF_{n-1}\right)^{2i} \left(2F_n^2 + \left(-1\right)^n\right)^{m-3i} \qquad (8)
\end{aligned}$$

and

$$\begin{aligned}
&L_{n+1}^{2m} + L_n^{2m} + L_{n-1}^{2m} \\
&= \sum_{i=0}^{\lfloor m/3 \rfloor} \binom{m-i}{m-3i}_{3,2} \left(L_{n+1}L_nL_{n-1}\right)^{2i} \left(L_{n+1}^2 - L_nL_{n-1}\right)^{m-3i} \\
&= \sum_{i=0}^{\lfloor m/3 \rfloor} \binom{m-i}{m-3i}_{3,2} \left(L_{n+1}L_nL_{n-1}\right)^{2i} \left(2L_n^2 - 5\left(-1\right)^n\right)^{m-3i}. \qquad (9)
\end{aligned}$$

The case $m = 1, 2$ and 3 coincides with Carlitz, Hunter and Koshy's propositions.

Now, we give modular identities belongs to the sequence Fibonacci and Lucas without proofs.

Corollary 1. *For $m, n \geq 0$, the following identities hold*

$$E_{n+1}^{2m+1} - E_n^{2m+1} - E_{n-1}^{2m+1} \equiv 0 \pmod{E_{n+1}E_nE_{n-1}}$$

and

$$E_{n+1}^{2m} + E_n^{2m} + E_{n-1}^{2m} \equiv 2\left(E_{n+1}^2 - E_nE_{n-1}\right)^m \pmod{E_{n+1}E_nE_{n-1}}$$

where $\{E_n\}_{n\geq 0}$ is Fibonacci or Lucas sequence.

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