On the order-$k$ generalized Lucas numbers

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Abstract

In this paper we give a new generalization of the Lucas numbers in matrix representation. Also we present a relation between the generalized order-$k$ Lucas sequences and Fibonacci sequences.

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1. Introduction

It is well known that $\{F_n\}$, the Fibonacci sequence is defined by a recurrence relation, that is, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ such that $F_1 = 1$ and $F_2 = 1$. Also, one can obtain the Fibonacci sequence by matrix methods. Indeed, it is clear that

$$
\begin{bmatrix}
F_n \\
F_{n+1}
\end{bmatrix} = A^n \begin{bmatrix}
0 \\
1
\end{bmatrix},
$$

where

$$
A = \begin{bmatrix}
0 & 1 \\
1 & 1
\end{bmatrix}.
$$

Kalman [1] mentioned that this is a special case of a sequence which is defined recursively as a linear combination of the preceding $k$ terms:

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\[ a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}, \]

where \( c_0, c_1, \ldots, c_{k-1} \) are real constants. In [1], Kalman obtained a number of closed-form formulas for the generalized sequence by matrix method.

In [2] Er defined \( k \) sequences of the generalized order - \( k \) Fibonacci numbers as shown: for \( n > 0 \) and \( 1 \leq i \leq k \)

\[ g_i^n = \sum_{j=1}^{k} c_j g_{n-j}^i \] (1)

with boundary conditions for \( 1 - k \leq n \leq 0 \),

\[ g_i^n = \begin{cases} 1 & \text{if } i = 1 - n, \\ 0 & \text{otherwise}, \end{cases} \] (2)

where \( c_j, 1 \leq j \leq k \), are constant coefficients, \( g_i^n \) is the \( n \)th term of \( i \)th sequence.

Er showed that

\[
\begin{bmatrix}
g_{n+1}^i \\
g_n^i \\
\vdots \\
g_{n-k+2}^i
\end{bmatrix}
= A
\begin{bmatrix}
g_n^i \\
g_{n-1}^i \\
\vdots \\
g_{n-k+1}^i
\end{bmatrix},
\] (3)

where

\[
A = 
\begin{bmatrix}
c_1 & c_2 & c_3 & \cdots & c_{k-1} & c_k \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix},
\] (4)

being a \( k \times k \) companion matrix. Then he derived

\[ G_{n+1} = AG_n, \] (5)

where

\[
G_n = 
\begin{bmatrix}
g_1^1 & g_2^1 & \cdots & g_k^1 \\
g_1^2 & g_2^2 & \cdots & g_k^2 \\
\vdots & \vdots & \ddots & \vdots \\
g_1^n & g_2^n & \cdots & g_k^n \\
g_1^{k+1} & g_2^{k+1} & \cdots & g_k^{k+1}
\end{bmatrix},
\] (6)

generalizing the matrix equation in (3).

Recently, Karaduman [3] showed that \( G_1 = A \), and therefore, \( G_n = A^n \). Also he proved that
In this paper we give an order-$k$ generalization of the usual Lucas sequence \{\(L_n\)\}, which is defined recursively as \(L_n = L_{n-1} + L_{n-2}\) for \(n \geq 3\) with initial condition \(L_1 = 1\) and \(L_2 = 3\). We present a matrix representation of the order-$k$ generalization of the Lucas sequence. Then we calculate the determinant of the matrix obtained by our sequence. Furthermore, we find a relation between \(G_n\) obtained by the sequence of the generalized order-$k$ Fibonacci numbers and the matrix obtained by the sequence of the generalized order-$k$ Lucas numbers.

2. The main results

Define \(k\) sequences of the generalized order-$k$ Lucas numbers as shown:

\[
l_i^n = \sum_{j=1}^{k} l_{n-j}^i,
\]

for \(n > 0\) and \(1 \leq i \leq k\), with boundary (initial) conditions

\[
l_i^n = \begin{cases} 
2 & \text{if } i = 2 - n, \\
-1 & \text{if } i = 1 - n, \\
0 & \text{otherwise,}
\end{cases}
\]

for \(1 - k \leq n \leq 0\), where \(l_i^n\) is the \(n\)th term of the \(i\)th sequence.

When \(i = 1\) and \(k = 2\), the generalized order-$k$ Lucas sequence reduces to negative usual Fibonacci sequence, i.e., \(l_i^n = -F_{n+1}\) for all \(n \in \mathbb{Z}^+\).

When we choose \(k = 4\) and \(i = 3\), the generalized order-$k$ Lucas sequence is as follows:

\[
\ldots, l_2^3 = -1, l_1^3 = 2, l_0^3 = 0, l_1^3 = 1, l_2^3 = 2, l_3^3 = 5, l_4^3 = 8, l_5^3 = 16, l_6^3 = 31, \ldots
\]

By (7), we can write

\[
\begin{bmatrix}
l_{n+1}^l \\
l_n^l \\
l_{n-1}^l \\
\vdots \\
l_{n-k+2}^l
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0
\end{bmatrix}
\begin{bmatrix}
l_n^l \\
l_{n-1}^l \\
l_{n-2}^l \\
\vdots \\
l_{n-k+1}^l
\end{bmatrix},
\]

\[\text{(9)}\]
for the generalized order-\(k\) Lucas sequences. Letting

\[
A = \begin{bmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0
\end{bmatrix},
\]

(10)

To deal with the \(k\) sequences of the generalized order-\(k\) Lucas sequences simultaneously, we define a \(k \times k\) matrix \(H_n\) as follows:

\[
H_n = \begin{bmatrix}
l^1_n & l^2_n & \ldots & l^k_n \\
l^1_{n-1} & l^2_{n-1} & \ldots & l^k_{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
l^1_{n-k+1} & l^2_{n-k+1} & \ldots & l^k_{n-k+1}
\end{bmatrix}.
\]

(11)

Generalizing Eq. (9), we have

\[
H_{n+1} = A \cdot H_n.
\]

(12)

**Lemma 1.** Let \(A\) and \(H_n\) be as in (10) and (11), respectively. Then \(H_{n+1} = A^n \cdot H_1\), where

\[
H_1 = \begin{bmatrix}
-1 & 1 & 1 & \ldots & 1 & 1 \\
-1 & 2 & 0 & \ldots & 0 & 0 \\
0 & -1 & 2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 2 & 0 \\
0 & 0 & 0 & \ldots & -1 & 2
\end{bmatrix}.
\]

(13)

**Proof.** By (12), we have \(H_{n+1} = A H_n\). Then by induction and a property of matrix multiplication, we have

\[
H_{n+1} = A^n \cdot H_1.
\]

Also \(H_1 = A \cdot K\), where

\[
K = \begin{bmatrix}
-1 & 2 & 0 & \ldots & 0 \\
0 & -1 & 2 & \ldots & 0 \\
0 & 0 & -1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & -1
\end{bmatrix}.
\]

Thus, \(H_{n+1} = A^{n+1} \cdot K\). \(\square\)
Theorem 2. Let \( H_n \) be as in (11). Then

\[
\det H_{n+1} = \begin{cases} 
-1 & \text{if } k \text{ is odd}, \\
(-1)^{n+1} & \text{if } k \text{ is even}.
\end{cases}
\]

Proof. From Lemma 1, we have \( H_{n+1} = A^{n+1} \cdot K \). Then

\[
\det H_{n+1} = (\det A)^{n+1} \cdot \det K,
\]

where

\[
\det A = (-1)^{k+1} \quad \text{and} \quad \det K = (-1)^k.
\]

Thus

\[
\det H_{n+1} = \begin{cases} 
-1 & \text{if } k \text{ is odd}, \\
(-1)^{n+1} & \text{if } k \text{ is even}.
\end{cases}
\]

The following theorem presents a relation between the generalized order-\( k \) Lucas sequence and Fibonacci sequence, which was given by Er in [2].

Theorem 3. Let \( G_n \) and \( H_n \) be as in (6) and (11), respectively. Then \( H_n = G_n \cdot K \), where \( K \) is the \( k \times k \) matrix in (13).

Proof. In [2] Er, showed that \( G_n = A^n \). Also we obtain \( H_n = A^n \cdot K \). Thus we have

\[
H_n = G_n \cdot K.
\]

In Theorem 3, letting \( k = 2 \). Then we have

\[
\begin{bmatrix}
l_n^1 & l_n^2 \\
l_{n-1}^1 & l_{n-1}^2
\end{bmatrix} = \begin{bmatrix}
g_n^1 & g_n^2 \\
g_{n-1}^1 & g_{n-2}^2
\end{bmatrix} \begin{bmatrix}
-1 & 2 \\
0 & -1
\end{bmatrix}.
\]

Therefore, \( l_n^2 = 2g_n^1 - g_n^2 \). Since \( g_n^1 = g_{n+1}^2 \) for all \( n \in \mathbb{Z} \), we have \( l_n^2 = 2g_{n+1}^2 - g_n^2 \), where \( l_n^2 \) and \( g_n^2 \) is the usual Lucas and Fibonacci numbers, respectively.

Indeed, we generalize a relation Lucas and Fibonacci numbers, i.e., \( L_n = 2F_{n+1} - F_n \) (see p. 176, [4]).

References