A GENERALIZED FILBERT MATRIX

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Abstract. A generalized Filbert matrix is introduced, sharing properties of the Hilbert matrix and Fibonacci numbers. Explicit formulæ are derived for the LU-decomposition, their inverses, and the Cholesky factorization. The approach is to use $q$-analysis and to leave the justification of the necessary identities to the $q$-version of Zeilberger’s celebrated algorithm.

1. Introduction

The Filbert matrix $H_n = (h_{ij})_{i,j=1}^n$ is defined by $h_{ij} = \frac{1}{F_{i+j-1}}$ as an analogue of the Hilbert matrix where $F_n$ is the $n$th Fibonacci number. It has been defined and studied by Richardson [3].

In this paper we will study the generalized matrix with entries $\frac{1}{F_{i+j+r}}$, where $r \geq -1$ is an integer parameter. The size of the matrix does not really matter, and we can think about an infinite matrix $\mathcal{F}$ and restrict it whenever necessary to the first $n$ rows resp. columns and write $\mathcal{F}_n$.

Our approach will be as follows. We will use the Binet form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{1 - q^n}{1 - q},$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = i/\sqrt{q}$. All the identities we are going to derive hold for general $q$, and results about Fibonacci numbers come out as corollaries for the special choice of $q$.

Throughout this paper we will use the following notations: $(x; q)_n = (1 - x)(1 - xq) \ldots (1 - xq^{n-1})$ and the Gaussian $q$-binomial coefficients

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}.$$

Furthermore, we will use Fibonomial coefficients

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \frac{F_n F_{n-1} \ldots F_{n-k+1}}{F_1 \ldots F_k}.$$

The link between the two notations is

$$\left\{ \begin{array}{c} n \\ k \end{array} \right\} = \alpha^{k(n-k)} \left[ \begin{array}{c} n \\ k \end{array} \right] \quad \text{with} \quad q = -\alpha^{-2}.$$

We will obtain the LU-decomposition $\mathcal{F} = L \cdot U$:

**Theorem 1.1.** For $1 \leq d \leq n$ we have

$$L_{n,d} = q^{\frac{n+d}{2}} i^{n+d} (-1)^n \left[ \begin{array}{c} n-1 \\ d-1 \end{array} \right] \left[ \begin{array}{c} 2d + r \\ d \end{array} \right] \left[ \begin{array}{c} n+d+r \\ d \end{array} \right]^{-1}$$

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Theorem 1.2. For $1 \leq d \leq n$ we have

$$U_{d,n} = q^{\frac{n-d}{2} + d + r} \left[ 2d + r - 1 \right]^{-1} \left[ n + d + r \right]^{-1} \left[ \frac{n}{d} \right] \frac{1 - q}{1 - q^n}$$

and its Fibonacci corollary

$$U_{d,n} = (-1)^{r(d+1)} \left[ 2d + r - 1 \right]^{-1} \left[ n + d + r \right]^{-1} \left[ \frac{n}{d} \right] \frac{1}{F_n}.$$

We could also determine the inverses of the matrices $L$ and $U$:

Theorem 1.3. For $1 \leq d \leq n$ we have

$$L_{n,d}^{-1} = q^{\frac{n-d}{2} + d + r} \left[ 2d + r - 1 \right]^{-1} \left[ n + d + r \right]^{-1} \left[ \frac{n}{d} \right] \frac{1 - q}{1 - q^n}.$$

and its Fibonacci corollary

$$L_{n,d}^{-1} = (-1)^{(n+1)d + \frac{n(n+1)}{2} + \frac{d(d+1)}{2}} \left[ n + r \right] \left[ n + d - 1 + r \right] \left[ \frac{n}{d} \right] \frac{1 - q}{1 - q^n}.$$

Theorem 1.4. For $1 \leq d \leq n$ we have

$$U_{d,n}^{-1} = q^{\frac{n^2 + d^2 + r + 1}{2} - (d+r)n + d - 1 + r} (-1)^{n-d} \left[ 2n + r \right] \left[ n + d + r - 1 \right] \left[ \frac{n}{d} \right] \frac{1 - q^n}{1 - q}.$$

and its Fibonacci corollary

$$U_{d,n}^{-1} = (-1)^{\frac{n(n+1)}{2} + \frac{d(d+1)}{2} - dn - rn + r} \left[ 2n + r \right] \left[ n + d - 1 + r \right] \left[ \frac{n}{d} \right] \frac{1}{F_n}.$$

As a consequence we can compute the determinant of $\mathcal{F}_n$, since it is simply evaluated as $U_{1,1} \cdots U_{n,n}$ (we only state the Fibonacci version):

Theorem 1.5.

$$\det \mathcal{F}_n = (-1)^{\frac{rn(n+1)}{2}} \prod_{d=1}^{n} \left[ 2d + r - 1 \right]^{-1} \left[ \frac{2d + r}{d} \right]^{-1} \frac{1}{F_d}.$$

Now we determine the inverse of the matrix $\mathcal{F}$. This time it depends on the dimension, so we compute $(\mathcal{F}_n)^{-1}$:

Theorem 1.6. For $1 \leq i, j \leq n$:

$$(\mathcal{F}_n)^{-1} = q^{\frac{r^2 + d^2 + r + 1}{2} - (i+j+r)n + i + j + r + 1} (-1)^{i+j+1} \times \left[ n + r + i \right] \left[ n + r + j \right] \left[ n - 1 \right] \left[ \frac{n - 1}{i} \right] \left[ n - 1 \right] \left[ \frac{n - 1}{j} \right] \frac{(1 - q^n)^2}{(1 - q^{r+i+j})(1 - q)}$$

and its Fibonacci corollary
Theorem 1.7. For \( n \geq d \):

\[
\mathcal{C}_{n,d} = (-1)^n q^{n+r+\frac{1}{2}d + \frac{d^2}{4}} \frac{\sqrt{1 - q^{2d+r}}(1 - q)}{1 - q^{2n+r}} \left[ \begin{array}{c} 2n + r \\ n-d \end{array} \right] \left[ \begin{array}{c} 2n + r - 1 \\ n-1 \end{array} \right]^{-1}
\]

and its Fibonacci corollary

\[
\mathcal{C}_{n,d} = (-1)^{\frac{d(d-1)}{2} + \frac{r(d+1)}{2}} \sqrt{\frac{F_{2d+r}}{F_{2n+r}}} \left[ \begin{array}{c} 2n + r \\ n-d \end{array} \right] \left[ \begin{array}{c} 2n + r - 1 \\ n-1 \end{array} \right]^{-1}.
\]

Notice that for odd \( r \), even the Fibonacci version may contain complex numbers.

2. Proofs

In order to show that indeed \( \mathcal{F} = L \cdot U \), we need to show that for any \( m, n \):

\[
\sum_d L_{m,d} U_{d,n} = \mathcal{F}_{m,n} = \alpha^{-m-n-r+1} \frac{1-q}{1-q^{m+n+r}}.
\]

In rewritten form the formula to be proved reads

\[
\sum_d (q^{d^2+(r-1)d-r} - q^{d^2+(r+1)d}) \left[ \begin{array}{c} 2m + r \\ m-d \end{array} \right] \left[ \begin{array}{c} 2n + r \\ n-d \end{array} \right]
= \frac{(1-q^{2n+r})(1-q^{2m+r})}{1-q^{m+n+r}} \left[ \begin{array}{c} 2m + r - 1 \\ m-1 \end{array} \right] \left[ \begin{array}{c} 2n + r - 1 \\ n-1 \end{array} \right].
\]

Nowadays, such identities are a routine verification using the \( q \)-Zeilberger algorithm, as described in the book [2]. We used Zeilberger’s own version [6], which is a Maple program. Mathematica users would get the same results using a package called \( q \text{Zeil} \) [5].

For interest, we also state (as a corollary) the corresponding Fibonacci identity:

\[
\sum_d (-1)^{r(d-1)} F_{2d+r} \left[ \begin{array}{c} 2m + r \\ m-d \end{array} \right] \left[ \begin{array}{c} 2n + r \\ n-d \end{array} \right] = \frac{F_{2n+r} F_{2m+r}}{F_{m+n+r}} \left[ \begin{array}{c} 2m + r - 1 \\ m-1 \end{array} \right] \left[ \begin{array}{c} 2n + r - 1 \\ n-1 \end{array} \right].
\]

Now we move to the inverse matrices. Since \( L \) and \( L^{-1} \) are lower triangular matrices, we only need to look at the entries indexed by \( (m, n) \) with \( m \geq n \):

\[
\sum_{n \leq d \leq m} L_{m,d} L_{d,n}^{-1}
= \sum_{n \leq d \leq m} q^{\frac{m-n}{2} + m+d} (-1)^m \left[ \begin{array}{c} m-1 \\ d-1 \end{array} \right] \left[ \begin{array}{c} 2d + r \\ d \end{array} \right] \left[ \begin{array}{c} m + d + r \\ d \end{array} \right]^{-1}.
\]
\[
\times q^{\frac{(n,q)^2}{2}} i^{n+d} (-1)^n \begin{bmatrix} d + r \\ n + r \end{bmatrix} n + d - 1 + r \right) [2d - 1 + r]^{-1} \\
= \frac{1}{1 - q^{2m+r}} \begin{bmatrix} 2m + r - 1 \\ m - 1 \end{bmatrix}^{-1} i^{m+n} (-1)^{m+n} \\
\times \sum_{n \leq d \leq m} q^{\frac{m-d+2(n-d)^2}{2}} (1 - q^{2d+r})(-1)^d \begin{bmatrix} 2m + r \\ m - d \end{bmatrix} \begin{bmatrix} n + d - 1 + r \\ n + r \end{bmatrix} [d - 1].
\]

The \( q \)-Zeilberger algorithm can evaluate the sum, and it is indeed \( [m = n] \), as predicted.

The argument for \( U \cdot U^{-1} \) is similar:

\[
\sum_{m \leq d \leq n} U_{m,d} U_{d,n}^{-1} \\
= (-1)^{m+n+q} q - \frac{2}{2} + m^2 + rm - \frac{2}{2} - rn \begin{bmatrix} 2m + r - 1 \\ m - 1 \end{bmatrix}^{-1} \begin{bmatrix} 2n + r - 1 \\ n - 1 \end{bmatrix} \frac{1 - q^{2n+r}}{1 - q^{n+m+r}} \\
\times \sum_{m \leq d \leq n} (-1)^d q^{\frac{d(d+1)}{2} - dn} \begin{bmatrix} n + d + r - 1 \\ d - m \end{bmatrix} \begin{bmatrix} n + m + r \\ n - d \end{bmatrix}.
\]

Again, the \( q \)-Zeilberger algorithm evaluates this to \( [m = n] \).

Now we turn to the inverse matrix:

\[
((T_n)^{-1} T_n)_{i,k} = i^{-k} (-1)^i q^{\frac{2i}{2} - (i+r)n+r} (1 - q^n)^2 \begin{bmatrix} n + r + i \\ n \end{bmatrix} \begin{bmatrix} n - 1 \\ i - 1 \end{bmatrix} \\
\times \sum_{j=1}^{n} q^{\frac{j(j+1)}{2} - jn} (-1)^i \begin{bmatrix} n + r + j \\ n \\ j - 1 \end{bmatrix} \frac{1}{(1 - q^{j+k+r})(1 - q^{r+i+j})}.
\]

And the \( q \)-Zeilberger algorithm evaluates this again to \( [i = k] \).

The Cholesky verification goes like this:

\[
\min_{m,n} \sum_{d=1}^{\min(m,n)} \hat{C}_{m,d} \hat{C}_{n,d} \\
= (-1)^{m+n+r+1} q^{\frac{m+n+r-1}{2}} \frac{1 - q}{1 - q^{2m+r} (1 - q^{2n+r})} \\
\times \begin{bmatrix} 2n + r - 1 \\ n - 1 \end{bmatrix}^{-1} \begin{bmatrix} 2m + r - 1 \\ m - 1 \end{bmatrix}^{-1} \sum_{d} q^{d(d+1)+rd} (1 - q^{2d+r}) \begin{bmatrix} 2m + r \\ m - d \end{bmatrix} \begin{bmatrix} 2n + r \\ n - d \end{bmatrix}.
\]

And again the \( q \)-Zeilberger algorithm evaluates this to

\[
\frac{1 - q}{1 - q^{m+n+r+1} q^{\frac{m+n+r-1}{2}}},
\]

as it should.

**References**


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