NEGATIVELY SUBSCRIPTED FIBONACCI AND LUCAS NUMBERS AND THEIR COMPLEX FACTORIZATIONS

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ABSTRACT. In this paper, we find families of \((0, -1, 1)\) – tridiagonal matrices whose determinants and permanents equal to the negatively subscripted Fibonacci and Lucas numbers. Also we give complex factorizations of these numbers by the first and second kinds of Chebyshev polynomials.

1. Introduction

The well-known Fibonacci sequence, \(\{F_n\}\), is defined by the recurrence relation, for \(n \geq 2\)
\[ F_{n+1} = F_n + F_{n-1} \] (1.1)
where \(F_1 = F_2 = 1\). The Lucas Sequence, \(\{L_n\}\), is defined by the recurrence relation, for \(n \geq 2\)
\[ L_{n+1} = L_n + L_{n-1} \] (1.2)
where \(L_1 = 1, L_2 = 3\).

Rules (1.1) and (1.2) can be used to extend the sequence backward, respectively, thus
\[ F_{-1} = F_1 - F_0, \quad F_{-2} = F_0 - F_{-1} \]
\[ L_{-1} = L_1 - L_0, \quad L_{-2} = L_0 - L_{-1}, \ldots, \]
and so on. Clearly
\[ F_{-n} = F_{-n+2} - F_{-n+1} = (-1)^{n+1} F_n, \] (1.3)
\[ L_{-n} = L_{-n+2} - L_{-n+1} = (-1)^n L_n. \] (1.4)

In [9] and [5], the authors give complex factorizations of the Fibonacci numbers by considering the roots of Fibonacci polynomials as follows
\[ F_n = \prod_{k=1}^{n-1} \left( 1 - 2i \cos \frac{\pi k}{n} \right), \quad n \geq 2. \] (1.5)
In [10] and [11], the authors establish the following forms:

\[ F_n = i^{n-1} \frac{\sin(n \cos^{-1}(-\frac{i}{2}))}{\sin(\cos^{-1}(-\frac{i}{2}))}, \quad n \geq 1 \quad (1.6) \]

\[ L_n = 2i^n \cos\left(n \cos^{-1}\left(-\frac{i}{2}\right)\right), \quad n \geq 1. \]

In [3], the authors prove (1.5) by considering how to the Fibonacci numbers can be connected to Chebyshev polynomials by determinants of a sequence of matrices, and then show that a connection between the Lucas numbers and Chebyshev polynomials by using a slightly different sequence of matrices as follows

\[ L_n = \prod_{k=1}^{n} \left(1 - 2i \cos\left(\frac{\pi}{n}\frac{k - \frac{1}{2}}{n}\right)\right), \quad n \geq 1. \]

There are many connections between permanents or determinants of tridiagonal matrices and the Fibonacci and Lucas numbers. For example, Minc [12] defines a \( n \times n \) super diagonal matrix \( F(n, k) \) for \( n > k \geq 2 \), and shows that the permanent of \( F(n, k) \) equals the generalized order-\( k \) Fibonacci numbers. In [14], the author gives the same result of Minc by the same matrix \( F(n, k) \) and using different a computing method of permanent, contraction. In particular, when \( k = 2 \), the matrix \( F(n, k) \) is reduced to the tridiagonal toeplitz matrix

\[ F(n, 2) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & \ddots \\ \ddots & \ddots & 1 \\ 0 & 1 & 1 \end{bmatrix} \]

and \( \text{per} F(n, 2) = F_{n+1}. \)

In [15], Lehmer proves a very general result on permanents of tridiagonal matrices whose main diagonal and super-diagonal elements are ones and whose subdiagonal entries are somewhat arbitrary.

Also in [16] and [17], the authors define the \( n \times n \) tridiagonal matrix \( M_n \) and show that the determinant of \( M(n) \) is the Fibonacci number \( F_{2n+2}. \) In [2] and [3], the authors define the \( n \times n \) tridiagonal matrix \( H(n) \) and show that the determinant of \( H(n) \) is the Fibonacci number \( F_n. \) In a similar family of matrices, the \((1, 1)\) element of \( H(n) \) is replaced with a 3, thus the determinants, [18], now generate the Lucas sequence \( L_n. \)

Recently, in [7], the authors find families of square matrices such that (i) each matrix is the adjacency matrix of a bipartite graph; and (ii) the permanent of the matrix is a sum of consecutive Fibonacci or Lucas numbers. Also, in [8], the authors define two tridiagonal matrices and then give the
relationships between the permanents and determinants of these matrices, and the terms of second order linear recurrences.

In this paper, we consider negatively subscripted Fibonacci and Lucas numbers and find associated families of tridiagonal matrices whose determinants or permanents equal to these numbers. Then we give the complex factorizations of these numbers by Chebyshev polynomial.

The permanent of an $n$-square matrix $A = (a_{ij})$ is defined by

$$\text{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where the summation extends over all permutations $\sigma$ of the symmetric group $S_n$.

Also one can find more applications of permanents in [13].

A matrix is said to be a $(-1, 0, 1)$-matrix if each of its entries are $1$, $0$ or 1.

Let $A = [a_{ij}]$ be an $m \times n$ real matrix row vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$. We say A is contractible on column (resp. row) $k$ if column (resp. row) $k$ contains exactly two nonzero entries. Suppose $A$ is contractible on column $k$ with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m - 1) \times (n - 1)$ matrix $A_{ij,k}$ obtained from $A$ by replacing row $i$ with $a_{jk}\alpha_i + a_{ik}\alpha_j$ and deleting row $j$ and column $k$ is called the contraction of $A$ on column $k$ relative to rows $i$ and $j$. If $A$ is contractible on row $k$ with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k,ij} = \left[ A_{ij,k}^T \right]^T$ is called the contraction of $A$ on row $k$ relative to columns $i$ and $j$. Every contraction used in this paper will be on the first column using the first and second rows. We say that $A$ can be contracted to a matrix $B$ if either $B = A$ or exist matrices $A_0, A_1, \ldots, A_t$ ($t \geq 1$) such that $A_0 = A$, $A_t = B$, and $A_r$ is a contraction of $A_{r-1}$ for $r = 1, 2, \ldots, t$.

Let us consider the following result (see [1]): Let $A$ be a nonnegative integral matrix of order $n > 1$ and let $B$ be a contraction of $A$. Then

$$\text{per} A = \text{per} B. \quad (1.7)$$

2. Negatively Subscripted Fibonacci and Lucas Numbers

In this section, we define families of tridiagonal matrices and then show that the determinants and permanents of these matrices equal to the negatively subscripted Fibonacci and Lucas numbers.

We start with negatively subscripted Fibonacci numbers. Now we define a $n \times n$ tridiagonal toeplitz $(0, -1, 1)$-matrix $A_n = [a_{ij}]$ with $a_{ii} = -1$ for
\[ 1 \leq i \leq n, \ a_{i,i+1} = a_{i+1,i} = 1 \text{ for } 1 \leq i \leq n - 1 \text{ and } 0 \text{ otherwise.} \]

That is,

\[
A_n = \begin{bmatrix}
-1 & 1 & 0 \\
1 & -1 & \ddots \\
\vdots & \ddots & 1 \\
0 & 1 & -1
\end{bmatrix}.
\] (2.1)

Then we give following Theorem.

**Theorem 1.** Let the matrix \( A_n \) have the form (2.1). Then, for \( n \geq 1 \)

\[ \text{per} A_n = F_{-(n+1)} \]

where \( F_{-n} \) is the \( n \)th negatively subscripted Fibonacci number.

**Proof.** If \( n = 1 \), then \( \text{per} A_1 = \text{per} [-1] = F_{-2} = -1 \).

If \( n = 2 \), then

\[
A_2 = \begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix}
\]

and hence \( \text{per} A_2 = F_{-3} = 2 \).

Let \( A^p_n \) be \( p \)th contraction of \( A_n \), \( 1 \leq p \leq n - 2 \). From the definition of \( A_n \), the matrix \( A_n \) can be contracted on column 1 so that

\[
A^1_n = \begin{bmatrix}
2 & -1 \\
1 & -1 & 1 \\
\vdots & \ddots & 1 \\
1 & 1 & -1
\end{bmatrix}.
\]

Since the matrix \( A^1_n \) can be contracted on column 1 and \( F_{-4} = -3 \), \( F_{-3} = 2 \),

\[
A^2_n = \begin{bmatrix}
-3 & 2 \\
1 & -1 & 1 \\
1 & -1 & \ddots \\
\vdots & \ddots & 1 \\
1 & 1 & -1
\end{bmatrix} = \begin{bmatrix}
F_{-4} & F_{-3} \\
1 & -1 & 1 \\
1 & -1 & \ddots \\
\vdots & \ddots & 1 \\
1 & 1 & -1
\end{bmatrix}.
\]

Continuing this process, we obtain

\[
A^r_n = \begin{bmatrix}
F_{-(r+2)} & F_{-(r+1)} \\
1 & -1 \ddots \\
\vdots & \ddots & 1 \\
1 & 1 & -1
\end{bmatrix}.
\]
for $3 \leq r \leq n - 4$. Hence,

$$A_n^{n-3} = \begin{bmatrix} F^{-(n-1)} & F^{-(n-2)} & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

which, by contraction of $A_n^{n-4}$ on column 1, gives

$$A_n^{n-2} = \begin{bmatrix} F^{-(n-2)} - F^{-(n-1)} & F^{-(n-1)} \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} F^{-n} & F^{-(n-1)} \end{bmatrix}.$$ 

By the Eq. (1.7) and the definition of the negatively subscripted Fibonacci numbers, we obtain

$$\text{per} A_n = \text{per} A_n^{n-2} = F^{-(n-1)} - F^{-n} = F^{-(n+1)}.$$ 

So the proof is complete. \qed

Second, we define a $n \times n$ tridiagonal $(0, -1, 1)$ - matrix $B_n = [b_{ij}]$ with $b_{ii} = -1$ for $2 \leq i \leq n$, $b_{i,i+1} = b_{i+1,i} = 1$ for $1 \leq i \leq n - 1$, $b_{11} = -\frac{1}{2}$ and 0 otherwise. That is,

$$B_n = \begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ 1 & -1 & \ddots \\ \ddots & \ddots & 1 \\ 0 & 1 & -1 \end{bmatrix}. \quad (2.2)$$

Now we give following Theorem.

**Theorem 2.** Let the matrix $B_n$ has the form (2.2). Then

$$\text{per} B_n = \frac{L_n - n}{2}$$

where $L_n$ is the $n$th negatively subscripted Lucas number.

**Proof.** If $n = 1$, then

$$\text{per} B_1 = \text{per} \begin{bmatrix} -\frac{1}{2} \end{bmatrix} = L_{-1}/2 = -1/2.$$ 

If $n = 2$, then

$$B_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

and hence $\text{per} B_2 = L_{-2}/2 = 3/2.$
Let $B_p^n$ be $p$th contraction of $B_n$, $1 \leq p \leq n - 2$. From the definition of $B_n$, the matrix $B_n$ can be contracted on column 1 so that

$$B^1_n = \begin{bmatrix}
\frac{3}{2} & -\frac{1}{2} & 0 \\
1 & -1 & 1 \\
1 & -1 & \ddots & 1 \\
0 & \ddots & \ddots & 1 \\
\end{bmatrix}.$$

Since the matrix $B^1_n$ can be contracted on column 1 and $L_{-3} = -4$, $L_{-2} = 3,$

$$B^2_n = \begin{bmatrix}
-\frac{1}{2} & \frac{3}{2} & 1 \\
1 & -1 & \ddots \\
\ddots & \ddots & 1 \\
1 & -1 & \\
\end{bmatrix} = \begin{bmatrix}
\frac{L_{-3}}{2} & \frac{L_{-2}}{2} \\
1 & -1 \\
\ddots & \ddots & 1 \\
1 & -1 \\
\end{bmatrix}.$$ 

Continuing this process, we obtain

$$B^r_n = \begin{bmatrix}
\frac{L_{-(r+1)}}{2} & \frac{L_{-r}}{2} \\
1 & -1 \\
\ddots & \ddots & 1 \\
1 & -1 \\
\end{bmatrix}$$

for $3 \leq r \leq n - 4$. Hence,

$$B^{n-3}_n = \begin{bmatrix}
\frac{L_{-(n-3)}}{2} & \frac{L_{-(n-3)}}{2} & 0 \\
1 & -1 & 1 \\
0 & 1 & -1 \\
\end{bmatrix}$$

which, by contraction of $B^{n-4}_n$ on column 1, gives

$$B^{n-2}_n = \begin{bmatrix}
(L_{-(n-3)} - L_{-(n-2)})/2 & L_{-(n-2)}/2 \\
1 & -1 \\
-1 & \ddots \\
\end{bmatrix} = \begin{bmatrix}
L_{-(n-1)}/2 & L_{-(n-2)}/2 \\
1 & -1 \\
\end{bmatrix}.$$ 

By the Eq. (1.7) and the definition of the negatively subscripted Lucas numbers, we obtain

$$\text{per}B_n = \text{per}B^{n-2}_n = (L_{-(n-2)} - L_{-(n-1)})/2 = L_{-n}/2.$$ 

So the proof is complete. \qed

A matrix $A$ is called **convertible** if there is an $n \times n$ $(1, -1)$—matrix $H$ such that $\text{per}A = \det(A \circ H)$, where $A \circ H$ denotes the Hadamard product of $A$ and $H$. Such a matrix $H$ is called a **converter** of $A$. 
Let $S$ be a $(1, -1)$ matrix of order $n$, defined by

\[
S = \begin{bmatrix}
1 & 1 & \ldots & 1 & 1 \\
-1 & 1 & \ldots & 1 & 1 \\
1 & -1 & \ldots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \ldots & -1 & 1 \\
\end{bmatrix}.
\]

Then we have that $F_{-(n+1)} = \det(A_n \circ S)$ and $L_{-n}/2 = \det(B_n \circ S)$ where $F_{-n}$ and $L_{-n}$ are the $n$th negatively subscripted Fibonacci and Lucas number, respectively.

Let we denote the matrices $A_n \circ S$ and $B_n \circ S$ by $C_n$ and $D_n$, respectively. Thus

\[
C_n = \begin{bmatrix}
-1 & 1 & 0 \\
-1 & -1 & \ddots \\
\ddots & \ddots & 1 \\
0 & -1 & -1 \\
\end{bmatrix}
\]

and

\[
D_n = \begin{bmatrix}
-\frac{1}{2} & 1 & 0 \\
-1 & -1 & \ddots \\
\ddots & \ddots & 1 \\
0 & -1 & -1 \\
\end{bmatrix}
\]

Also it is clear that the value of following determinant is independent of $x$: (see p.105, [19])

\[
\begin{vmatrix}
a & x & 0 \\
\frac{1}{x} & a & \ddots \\
\ddots & \ddots & \ddots & x \\
0 & \frac{1}{x} & a \\
\end{vmatrix}.
\]

Using the above result and considering the following matrices

\[
\hat{C}_n = \begin{bmatrix}
-1 & -\sqrt{1} & 0 \\
\sqrt{1} & -1 & \ddots \\
\ddots & \ddots & \ddots & -\sqrt{1} \\
0 & \sqrt{1} & -1 \\
\end{bmatrix}
\]
and
\[
\hat{D}_n = \begin{bmatrix}
-1/2 & -\sqrt{1} & 0 \\
\sqrt{1} & -1 & \ddots \\
& \ddots & \ddots & -\sqrt{1} \\
0 & \sqrt{1} & -1
\end{bmatrix},
\]
we can write that
\[
\begin{align*}
\det \hat{C}_n &= \det C_n = \text{per} A_n = F_{-(n+1)}, \\
\det \hat{D}_n &= \det D_n = \text{per} B_n = L_{-n}/2.
\end{align*}
\]
Furthermore, from [13], we have that let \( A \) be a tridiagonal matrix, and let \( \tilde{A} = (\tilde{a}_{ij}) \) be defined by \( \tilde{a}_{st} = a_{st} \) if \( s \neq t \) and \( \tilde{a}_{ss} = a_{ss} \), for all \( s \) and \( t \) \((i = \sqrt{-1})\). Then we have
\[
\text{per} (A) = \det (\tilde{A}).
\]
Also let we define the following matrices;
\[
\hat{C}_n = \begin{bmatrix}
-1 & i & 0 \\
i & -1 & \ddots \\
& \ddots & \ddots & i \\
0 & i & -1
\end{bmatrix} \tag{2.3}
\]
and
\[
\hat{D}_n = \begin{bmatrix}
-1/2 & i & 0 \\
i & -1 & \ddots \\
& \ddots & \ddots & i \\
0 & i & -1
\end{bmatrix}. \tag{2.4}
\]
Thus we have following Corollaries without proof.

**Corollary 1.** Let the \( n \times n \) tridiagonal toeplitz matrix \( \hat{C}_n \) as in (2.3). Then, for \( n \geq 1 \)
\[
\det \hat{C}_n = F_{-(n+1)}.
\]

**Corollary 2.** Let the \( n \times n \) tridiagonal matrix \( \hat{D}_n \) be as in (2.4). Then, for \( n \geq 1 \)
\[
\det \hat{D}_n = L_{-n}/2.
\]
3. Complex Factorization of the Negatively Subscripted Fibonacci Numbers

In [3], the authors consider the relationships between the certain tridiagonal determinants, and, the usual Fibonacci and Lucas numbers. Then using the eigenvalues of these tridiagonal matrices, the authors give the complex factorizations of the usual Fibonacci and Lucas numbers. Following the method of [3], we find the eigenvalues of the two tridiagonal matrices whose determinants associated with the negatively subscripted Fibonacci and Lucas numbers. Therefore, we give the complex factorizations of the negatively subscripted Fibonacci and Lucas numbers.

There are variety of ways of computing matrix determinants (see [4] and [6] for more details). In addition to the method of cofactor expansion, the determinant of a matrix can be found by taking the product of its eigenvalues. Therefore, we will compute the spectrum of $C_n$ to find an alternative representation of $\det C_n$.

Now we define another $n \times n$ tridiagonal toeplitz matrix $V_n = [v_{ij}]$ with $v_{ii} = 0$ for $1 \leq i \leq n$ and $v_{i,i-1} = v_{i-1,i} = 1$ for $2 \leq i \leq n$ and 0 otherwise. Clearly

$$V_n = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \ddots \\ \ddots & \ddots & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$  \hspace{1cm} (3.1)

So it is clear that $\hat{C}_n = -I + iV_n$. Then we give following Theorem.

**Theorem 3.** Let $F_{-n}$ be the $n$th negatively subscripted Fibonacci number. Then, for $n \geq 1$

$$F_{-(n+1)} = \prod_{j=1}^{n} \left( -1 - 2i \cos \left( \frac{n j}{n+1} \right) \right).$$

**Proof.** Let $\lambda_j, j = 1, 2, \ldots, n$, be the eigenvalues of $V_n$ with respect to eigenvectors $x_i$. Then, for all $j$

$$\hat{C}_n x_j = (-I + iV_n) x_j = -Ix_j + iV_n x_j = -x_j + i\lambda_j x_j = (-1 + i\lambda_j) x_j.$$  \hspace{1cm} (3.2)

Therefore, $\mu_j = -1 + i\lambda_j, j = 1, 2, \ldots, n$, are the eigenvalues of $\hat{C}_n$. Hence, for $n \geq 1$

$$\det \hat{C}_n = \prod_{j=1}^{n} (-1 + i\lambda_j).$$  \hspace{1cm} (3.3)

To compute the $\lambda_j$’s, we recall that each $\lambda_j$ is a zero of the characteristic polynomial $p_n(\lambda) = |V_n - \lambda I|$. Since $V_n - \lambda I$ is a tridiagonal toeplitz
matrix, i.e.,

\[
V_n - \lambda I = \begin{pmatrix}
-\lambda & 1 & & \\
1 & -\lambda & & \\
& \ddots & \ddots & \\
& & 1 & -\lambda
\end{pmatrix},
\]  

(3.4)

we can establish a recursive formula for the characteristic polynomials \(V_n\):

\[
p_1 (\lambda) = -\lambda,
p_2 (\lambda) = \lambda^2 - 1,
p_n (\lambda) = -\lambda p_{n-1} (\lambda) - p_{n-2} (\lambda).
\]

(3.5)

This family of characteristic polynomials can be transformed into another family \(\{U_n (x), n \geq 1\}\) by taking \(\lambda \equiv -2x\):

\[
U_1 (x) = 2x,
U_2 (x) = 4x^2 - 1,
U_n (x) = 2xU_{n-1} (x) - U_{n-2} (x).
\]

(3.6)

The family \(\{U_n (x), n \geq 1\}\) is the set of Chebyshev polynomials of second kind. It is a well-known fact (see [10]) that defining \(x = \cos \theta\) allows the Chebyshev polynomials of the second kind to be written as:

\[
U_n (x) = \frac{\sin [(n + 1) \theta]}{\sin \theta}.
\]

(3.7)

From (3.7), we can see that the roots of \(U_n (x) = 0\) are given by \(\theta_k = \frac{\pi k}{n + 1}, k = 1, 2, \ldots, n\), or \(x_k = \cos \theta_k = \cos \frac{\pi k}{n + 1}, k = 1, 2, \ldots, n\). Applying the transformation \(\lambda \equiv -2x\), we have the eigenvalues of \(V_n\):

\[
\lambda_k = -2 \cos \left(\frac{\pi k}{n + 1}\right), k = 1, 2, \ldots, n.
\]

(3.8)

Considering Corollary 1, the Eqs. (3.3) and (3.8), we obtain

\[
F_{-(n+1)} = \det \hat{C}_n = \prod_{j=1}^{n} \left( -1 - 2i \cos \left( \frac{\pi j}{n + 1} \right) \right)
\]

which is desired. \(\square\)

**Theorem 4.** Let \(F_n\) be nth negatively subscripted Fibonacci number. Then, for \(n \geq 1\)

\[
F_{-(n+1)} = i^n \frac{\sin \left( (n + 1) \cos^{-1} \left( \frac{\pi}{2} \right) \right)}{\sin \left( \cos^{-1} \left( \frac{\pi}{2} \right) \right)}.
\]
Proof. From (3.4), we can think of Chebyshev polynomials of the second kind as being generated by determinants of successive matrices of the form

\[ K_n(x) = \begin{bmatrix} 2x & 1 \\ 1 & 2x \\ & \ddots & \ddots \\ & & 1 & 2x \\ & & & 1 \end{bmatrix}, \tag{3.9} \]

where \( K_n(x) \) is \( n \times n \). If we denote that \( \breve{C}_n = iK_n(\frac{i}{2}) \), then we obtain:

\[ \det \breve{C}_n = i^n \det K_n \left( \frac{i}{2} \right) = i^n U_n \left( \frac{i}{2} \right). \tag{3.10} \]

Combining the result of Corollary 1, the Eqs. (3.7) and (3.10) yields, for \( n \geq 1 \)

\[ F_{-(n+1)} = i^n \frac{\sin \left( (n+1) \cos^{-1} \left( \frac{1}{2} \right) \right)}{\sin \left( \cos^{-1} \left( \frac{1}{2} \right) \right)}. \]

So the proof is complete. \( \square \)

**Theorem 5.** Let \( L_{-n} \) be \( n \)th negatively subscripted Lucas number. Then, for \( n \geq 1 \)

\[ L_{-n} = \prod_{k=1}^{n} \left( -1 - 2i \cos \frac{\pi (k - \frac{1}{2})}{n} \right). \]

Proof. From Corollary 2, we have that \( 2 \det \breve{D}_n = L_{-n} \). We will not compute the spectrum of \( \breve{D}_n \) directly. Instead, we will note that the following:

\[ (\det (I + e_1 e_1^T)) = 2 \]

\[ \det \breve{D}_n = \frac{1}{2} \det \left( (I + e_1 e_1^T) \breve{D}_n \right), \tag{3.11} \]

where \( e_j \) is the \( j \)th column of the identity matrix. Thus we can write that the right-side of (3.11) as follows

\[ \frac{1}{2} \det \left( (I + e_1 e_1^T) \breve{D}_n \right) = \frac{1}{2} \det \left( -I + i (V_n + e_1 e_2^T) \right) \]

(3.12)

where the matrix \( V_n \) is given by (3.1). Let \( \gamma_j, j = 1, 2, \ldots, n, \) be the eigenvalues of \( V_n + e_1 e_2^T \) with respect to eigenvectors \( y_j \). Then, for all \( j \)

\[ (-I + i (V_n + e_1 e_2^T)) y_j = -I y_j + i (V_n + e_1 e_2^T) y_j = -y_j + i \gamma_j y_j = (-1 + i \gamma_j) y_j. \]

Thus

\[ \frac{1}{2} \det \left( -I + i (V_n + e_1 e_2^T) \right) = \frac{1}{2} \prod_{k=1}^{n} (-1 + i \gamma_j). \tag{3.13} \]
To compute the $\gamma_j$’s, we recall that all $\gamma$ is a zero of the characteristic polynomial $t_n(\gamma) = \det (V_n + e_1e_2^T - \gamma I)$. Since $\det (I - \frac{1}{2} e_1e_1^T) = \frac{1}{2}$, we can alternately write the characteristic polynomial as
\[ t_n(\gamma) = 2 \det \left[ (I - \frac{1}{2} e_1e_1^T) (V_n + e_1e_2^T - \gamma I) \right]. \] (3.14)
Since $t_n(\gamma)$ is twice the determinant of a tridiagonal matrix, that is,
\[ t_n(\gamma) = 2 \det \begin{bmatrix} -\frac{1}{2} & 1 & & & \\ 1 & -\gamma & \ddots & & \\ & \ddots & \ddots & 1 & \\ & & 1 & -\gamma & \end{bmatrix}, \] (3.15)
one can derive a recursive formula for $\frac{t_n(\gamma)}{2}$:
\[
\begin{align*}
\frac{t_1(\gamma)}{2} &= -\frac{1}{2} \\
\frac{t_2(\gamma)}{2} &= \frac{\gamma^2}{2} - 1 \\
\frac{t_n(\gamma)}{2} &= -\gamma t_{n-1}(\gamma) - t_{n-2}(\gamma).
\end{align*}
\] (3.17)
This family of polynomials can be transformed into another family $\{T_n(x), n \geq 1\}$ by taking $\gamma \equiv -2x$:
\[
\begin{align*}
T_1(x) &= x, \\
T_2(x) &= 2x^2 - 1, \\
T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x).
\end{align*}
\] The family $\{T_n(x), n \geq 1\}$ is the set of Chebyshev polynomials of first kind. In [10], Rivlin presents that defining $x \equiv \cos \theta$ allows the Chebyshev polynomials of the first kind to be written as
\[ T_n(x) = \cos n \theta. \] (3.16)
From the Eq. (3.16), one can see that the roots of $T_n(x) = 0$ are given by
\[
\theta_k = \frac{\pi (k - \frac{1}{2})}{n} \quad \text{or} \quad x_k = \cos \theta_k = \cos \frac{\pi (k - \frac{1}{2})}{n} \quad \text{for} \quad k = 1, 2, \ldots, n.
\]
Applying the transformation $\gamma \equiv -2x$ and considering the roots of the (3.14) are also roots of $\det (V_n + e_1e_2^T - \gamma I) = 0$, we have the eigenvalues of $V_n + e_1e_2^T$:
\[ \gamma_k = -2 \cos \frac{\pi (k - \frac{1}{2})}{n} \quad \text{for} \quad k = 1, 2, \ldots, n. \] (3.17)
From Corollary 2, the Eqs. (3.17) and (3.13), we obtain
\[ L_{-n} = \prod_{k=1}^{n} \left( -1 - 2i \cos \frac{\pi (k - \frac{1}{2})}{n} \right). \]
So the proof is complete. \qed
Theorem 6. Let $L_{-n}$ be $n$th negatively subscripted Lucas number. Then, for $n \geq 1$

$$L_{-n} = 2i^n \cos \left( n \cos^{-1} \left( \frac{i}{2} \right) \right).$$

Proof. From (3.15), we think of Chebyshev polynomials of the first kind as being generated by determinants of successive matrices of the form

$$G_n(x) = \begin{bmatrix}
x & 1 \\
1 & 2x \\
\vdots & \vdots \\
1 & 2x \\
\end{bmatrix}_{n \times n}.
$$

We note that $\det D_n = iG_n \left( \frac{i}{2} \right)$, thus

$$\det D_n = i^n \det G_n \left( \frac{i}{2} \right) = i^n T_n \left( \frac{i}{2} \right). \quad (3.18)$$

From Corollary 2, the Eqs. (3.16) and (3.18), we obtain

$$L_{-n} = 2i^n \cos \left( n \cos^{-1} \left( \frac{i}{2} \right) \right).$$

So the proof is complete. \qed

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