SOME SUBSEQUENCES OF THE GENERALIZED FIBONACCI AND LUCAS SEQUENCES

EMRAH KILIC

Abstract. We derive first-order nonlinear homogeneous recurrence relations for certain subsequences of generalized Fibonacci and Lucas sequences. We also present a polynomial representation for the terms of Lucas subsequence.

1. INTRODUCTION

Let $p$ and $q$ be nonzero integers such that $\Delta = p^2 - 4q \neq 0$. The generalized Fibonacci sequence $\{U_n(p, q)\}$, or briefly $\{U_n\}$, and Lucas sequence $\{V_n(p, q)\}$, or briefly $\{V_n\}$, are defined by for $n > 1$

$$U_n = pU_{n-1} - qU_{n-2} \quad \text{and} \quad V_n = pV_{n-1} - qV_{n-2},$$

where $U_0 = 0, U_1 = 1$ and $V_0 = 2, V_1 = p$, respectively.

When $p = -q = 1$, $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number).

The Binet forms of $\{U_n\}$ and $\{V_n\}$ are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n,$$

where $\alpha$ and $\beta$ are the roots of $x^2 - px + q = 0$.

In [5], the solution of the following first order cubic recursion was asked

$$a_{n+1} = 5a_n^3 - 3a_n, \quad a_0 = 1. \quad (1.1)$$

Then the solution was given as $a_n = F_{3^n}$ in [7]. After this, similarly the solution of recurrence

$$P_{n+1} = 25P_n^5 - 25P_n^3 + 5P_n, \quad P_0 = 1 \quad (1.2)$$

was also asked. Then the solution was given as $P_n = F_{5^n}$.
As an addendum to the solution of the problem given in [7], Klamkin asked the solutions of recurrences:

\[ A_{n+1} = A_n^2 - 2, \quad A_1 = 3, \]
\[ B_{n+1} = B_n^4 - 4B_n^2 + 2, \quad B_1 = 7, \]
\[ C_{n+1} = C_n^6 - 6C_n^4 + 9C_n^2 - 2, \quad C_1 = 18. \]

Then the solutions of them were given as

\[ A_n = L_{2^n}, \quad B_n = L_{4^n} \text{ and } C_n = L_{6^n}. \]

In [1], the author give a recurrence relation for the Fibonacci subsequence \( \{F_{k_n}\} \) for positive odd \( k \), which generalize (1.1) and (1.2). In [2], some generalizations of the results of [1] were obtained for the sequences \( \{U_n(p, -1)\} \) and \( \{V_n(p, -1)\} \).

Meanwhile Prodinger [3] proved a general expansion formula for a sum of powers of Fibonacci numbers, as considered by Melham, as well as some extensions.

In this paper, we find first-order nonlinear recurrence relation for the subsequence \( \{U_{k_n}\} \) of generalized Fibonacci sequence \( \{U_n\} \) for odd \( k \), and first-order nonlinear recurrence relation for the subsequence \( \{V_{k_n}\} \) of generalized Lucas sequence \( \{V_n\} \) for both odd and even \( k \). We also give a polynomial representation for the generalized Lucas number \( V_{k_n} \) in terms of generalized Fibonacci numbers \( U_{k_n} \) of degree \( k \) for even \( k \).

2. Recurrence Relations

We find first-order nonlinear recursions for the sequences \( \{U_{k_n}\} \) and \( \{V_{k_n}\} \) for certain \( k \)'s. We need the following result for further use.

**Lemma 1.** For \( n, t \geq 0 \),

\[ i) \quad U_{(2t+1)n} = U_n \sum_{k=0}^{t} \frac{2t+1}{t+k+1} \frac{(t+k+1)}{2k+1} \Delta_k q^{n(t-k)} U_{n2^k}, \]
\[ ii) \quad V_{2tn} = \sum_{k=0}^{t} \frac{2t}{t+k} \Delta_k q^{n(t-k)} U_{n2^k}, \]
\[ iii) \quad V_{(2t+1)n} = V_n \sum_{k=0}^{t} (-1)^{t+k} \frac{2t+1}{t+k+1} \frac{(t+k+1)}{2k+1} \Delta_k q^{n(t-k)} V_{n2^k}, \]
\[ iv) \quad V_{2tn} = \sum_{i=0}^{t} (-1)^{t-i} \frac{2t}{t+i} \frac{(t+i)}{2i} \Delta_i q^{(t-i)n} V_{n2^i}. \]
Proof. In order to prove the claimed identities, it is sufficient to use the following well-known formulas (for more details, see [6]):

\[ X^m + Y^m = \sum_{k=0}^{[m/2]} (-1)^k \frac{m}{m-k} \binom{m-k}{k} (XY)^k (X + Y)^{m-2k}, \quad m \geq 1, \]

(2.1)

and

\[ X^m - Y^m \]

\[ \frac{X^m - Y^m}{X - Y} = \sum_{k=0}^{(m-1)/2} (-1)^k \binom{m-k-1}{k} (XY)^k (X + Y)^{m-2k-1}, \quad m \geq 1. \]

For example, the claim (iii) follows from by taking \( X = \alpha^n, \ Y = \beta^n \) and \( m = 2t \) in (2.1). The other claims are similarly obtained.

For odd \( k \), we give a first-order nonlinear recurrence relation for the sequence \( \{U_{k^n}\} \):

**Theorem 1.** For odd \( k > 0 \) and \( n \geq 0 \),

\[ U_{k^{n+1}} = \Delta \frac{k+1}{2} U_{k^n} + \sum_{i=0}^{(k-3)/2} \frac{2k}{k+2i+1} \binom{k+1}{2i+1} \Delta i q^{k^n} (\frac{k+1}{2} - i) U_{k^{2i+1}}. \]

Proof. From the Binet formula of \( \{U_n\} \) and by the binomial theorem, we obtain

\[ U_{k^n} = \left( \frac{\alpha^{k^n} - \beta^{k^n}}{\alpha - \beta} \right)^k \]

\[ = \frac{1}{\Delta^{(k-1)/2}} \left( U_{k^{n+1}} + \sum_{j=1}^{(k-1)/2} \binom{k}{j} (-1)^j \beta^j q^{k^n} (\Delta i q^{k^n} (\frac{k+1}{2} - i) U_{k^{2i+1}}) \right), \]

where \( \Delta \) is defined as before. By (2.2), we obtain for odd \( k \),

\[ U_{k^{n+1}} = \Delta \frac{k-1}{2} U_{k^n} - \sum_{j=1}^{(k-1)/2} \binom{k}{j} q^{k^n} (-1)^j U_{(k-2j)k^n}. \]

(2.3)

By (i) in Lemma 1 and (2.3), we conclude

\[ U_{k^{n+1}} = \Delta \frac{k-1}{2} U_{k^n} \]

\[ - \sum_{j=1}^{k-1} \sum_{i=0}^{(k-1)/2 - j} \binom{k}{j} \frac{2k}{k+2i+1} \Delta i q^{k^n} (\frac{k+1}{2} - i) U_{k^{2i+1}} \]

which, after reversing the summation order, can be rewritten as

\[ U_{k^{n+1}} = \Delta \frac{k-1}{2} U_{k^n} - \sum_{i=0}^{(k-1)/2} \Delta i q^{k^n} (\frac{k+1}{2} - i) A_{i,k} U_{k^{2i+1}}, \]

(2.4)

where

\[ A_{i,k} = \sum_{j=1}^{(k-1)/2} (-1)^j \binom{k}{j} \frac{2k}{k+2i+1} \Delta i q^{k^n} (\frac{k+1}{2} - i). \]
Since $A_{k+1/2,k} = 0$, the equality (2.4) becomes

$$U_{k+1} = \Delta^{(k-1)/2} U_{k} - \sum_{i=0}^{(k-3)/2} \Delta^i A_{i,k} q^k \left( \frac{k-1}{2} - i \right) U_{k+1}.$$  

From (pp. 58, [4]), it is known that for $1 \leq m \leq (k-3)/2$

$$\sum_{j=1}^{m} (-1)^j \frac{k-2j}{k-m-j} \frac{(k-m-j)}{m-j} = -\frac{k}{k-m} \binom{k-m}{m}.$$  

(2.5)

In order to obtain $A_{i,k}$ as $\frac{2k}{k+2i+1} \binom{k+1}{2i+1}$, it is sufficient to replace $m$ by $(k-1)/2 - i$ in (2.5). Thus we obtain the claimed result. \hfill \Box

For example, when $k = 7$, we have that

$$U_{7+1} = \Delta^2 U_{7} + 7 \Delta^2 q^7 U_{7}^5 + 14 \Delta q^7 U_{7}^3 + 7 q^7 U_{7}.$$  

We now give a nonlinear first order recurrence relation for the sequence $\{V_k\}$ for odd $k$.

**Theorem 2.** For $n > 0$ and odd $k > 1$,

$$V_{k+1} = V_k - \sum_{i=0}^{k-3} \binom{k-1}{2i+1} \frac{2k}{2i+1} (-1)^{i+\frac{k+1}{2}-i} q^k \left( \frac{k-1}{2} - i \right) V_{k+1}.$$  

Proof. It is easy to see that

$$V_k = \sum_{j=0}^{k} \binom{k}{j} \beta^{jk} \alpha^{(k-j)k} = V_{k+1} + \sum_{j=1}^{(k-1)/2} \binom{k}{j} q^{(k-2j)} V_{k}.$$  

By (iii) in Lemma 1, we write

$$V_{k+1} = V_k - \sum_{j=0}^{k-1} \sum_{i=0}^{k-j} \binom{k-1}{j} \left( \frac{k+1}{2i+1} \right) (k-2j) \left( \frac{k-1}{2} - i \right) (-1)^{i+\frac{k+1}{2}-i} q^k \frac{k-2j}{(k+1)(2i+1)} V_{k+1}$$  

which, by reversing the summation order, becomes

$$V_{k+1} = V_k - \sum_{i=0}^{k-1} \sum_{j=1}^{k-1} \binom{k-1}{j} \left( \frac{1}{2i+1} \right) (k+1-2i-j) \left( \frac{k-1}{2} - i \right) \left( \frac{k-1}{2} + j \right) \left( \frac{k+1}{2} - i \right) q^k \frac{k-2j}{(k+1)(2i+1)} V_{k+1}$$  

For the sum

$$\sum_{j=1}^{k-1} \binom{k-1}{j} \left( \frac{1}{2i+1} \right) \left( \frac{k+1}{2} - i \right) \left( \frac{k-1}{2} + j \right) \left( \frac{k+1}{2} - i \right)$$

if we take $m = (k-1)/2 - i$ in (2.5), we obtain the claimed result. \hfill \Box

We now give a nonlinear first order recurrence relation for the sequence $\{V_k\}$ for even $k$. 


Theorem 3. For \( n > 0 \) and even \( k > 1 \),

\[
V_{k^{n+1}} = V_{k^n} - \sum_{i=0}^{k/2 - 1} (-1)^i \frac{k}{2i} \binom{k/2 - 1}{i} \frac{2k}{2i - k} q^{(k/2 - 1)k^n} V_{k^{i}}.
\]

Proof. By the binomial theorem, we have that for even \( k \),

\[
V_{k^{n+1}} = V_{k^n} + \left(\frac{k}{k/2}\right) q^{k^{n+1}} - \sum_{j=1}^{k/2} \binom{k}{j} q^{k^n} V_{(k-j)k^n}.
\]

From (iv) in Lemma 1, we write

\[
V_{k^{n+1}} = V_{k^n} - \sum_{i=0}^{k/2 - 1} (-1)^i \frac{k}{2i} \binom{k/2 - 1}{i} \frac{2k}{2i - k} q^{(k/2 - 1)k^n} V_{k^{i}},
\]

as claimed.

For example, when \( k = 6 \), we have that

\[
V_{6^{n+1}} = V_{6^n} - 6q^{6^n} V_{6^n}^4 + 9q^{6^n} 2V_{6^n}^2 - 2q^{6^n} 3.
\]

3. A Polynomial Representation

We give a polynomial representation for the Lucas number \( V_{k^n} \) in terms of the generalized Fibonacci numbers \( U_{k^n} \) for even \( k \).

Theorem 4. For even \( k > 0 \), \( n \geq 0 \) and

\[
V_{k^{n+1}} = \sum_{i=0}^{k/2} \frac{2k}{k + 2i} \binom{k/2}{i} \frac{k}{2i} q^{k^n} U_{k^n} q^{k^n (k/2 - i)}.
\]

Proof. Consider

\[
U_{k^n} = \frac{1}{\Delta^{k/2}} \sum_{j=0}^{k} (-1)^j \beta^j \alpha^{(k-j)k^n}
\]

\[
= \frac{1}{\Delta^{k/2}} \left( V_{k^{n+1}} - (-1)^{k/2} q^{k^{n+1}} \frac{k}{k/2} + \sum_{j=1}^{k/2} (-1)^j \binom{k}{j} V_{(k-j)k^n} q^{jk^n} \right).
\]
By (ii) in Lemma 1 and reversing the summation order of the equation above, we write

\[ U_{kn}^k = \frac{1}{\Delta k/2} (V_{kn+1}^k + (-1)^{\frac{k}{2}} q^{kn+1/2} \binom{k}{k/2} + \sum_{j=1}^{\frac{k-2}{2}} \sum_{i=0}^{\frac{k-2}{2}} \binom{k}{j} \binom{k/2-j+i}{2i} \Delta^i q^{kn(k/2-i)} U_{kn}^{2i} ), \]

which becomes,

\[ = \frac{1}{\Delta k/2} \left( V_{kn+1}^k + \sum_{i=0}^{\frac{k-2}{2}} \sum_{j=1}^{\frac{k-2}{2}} (-1)^{j} \binom{k-2j}{k/2-j+i} \binom{k}{j} \binom{k/2-j+i}{2i} q^{kn(k/2-i)} \Delta^i U_{kn}^{2i} \right). \]

If we take \( m = \frac{k}{2} - i \) in (2.5) for \( 1 \leq m \leq k/2 \), the last equation takes the form:

\[ U_{kn}^k = \frac{1}{\Delta k/2} \left( V_{kn+1}^k - \sum_{i=0}^{\frac{k-2}{2}} \frac{2k}{k+2i} \binom{i+k/2}{2i} \Delta^i U_{kn}^{2i} q^{kn(k/2-i)} \right), \]

as claimed. \( \square \)

When \( k = 6 \), we get

\[ V_{6n+1}^6 = \Delta^3 V_{6n}^6 + 6 \Delta^2 V_{6n}^4 q^{6n} + 9 \Delta V_{6n}^2 q^{6n+2} + 2q^{6n+3}. \quad (3.1) \]

Notice that even the coefficients of the formula in (3.1) and (2.6) appears to be the terms of the sequence \( A034807 \) in the OEIS.

**Conclusions**

Throughout the paper, we obtain recurrence relations for the sequences \( \{U_{kn}\} \) and \( \{V_{kn}\} \) for certain \( k \)'s (not all \( k \)'s) and obtain a polynomial representation for the generalized Lucas number \( V_{kn} \) in terms of generalized Fibonacci numbers \( U_{kn} \) of degree \( k \) for even \( k \). In order to clear how the remaining cases could not be obtained, we note some facts here. Since we never reach at the statement \( U_{kn+1}^k \) when we expand the \( k \)th powers of the statements \( U_{kn} \) and \( V_{kn} \) by the binomial theorem for even integer \( k \), we can’t give a recurrence relation for \( U_{kn+1}^k \) for even \( k \). As a second remaining case, is there a polynomial representation of \( U_{kn}^k \) in terms of \( V_{kn} \) for odd \( k \) ? Related with this question, we note that while doing required operations, there is a problem (in reversing the summation order) so that we couldn’t find a representation for the term \( U_{kn+1}^k \) in terms of \( V_{kn} \).

**Acknowledgement**

The author would thank to the anonymous referee for his/her valuable suggestions.
REFERENCES