SOME CLASSES OF ALTERNATING WEIGHTED BINOMIAL SUMS

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Abstract. In this paper, we consider three classes of generalized alternating weighted binomial sums of the form
\[ \sum_{i=0}^{n} \binom{n}{i} (-1)^i f(n, i, k, t) \]
where \( f(n, i, k, t) \) will be chosen as \( U_{kti}V_{kn-kt} \) and \( U_{kti}V_{k(t+1)n-(k+2)t} \). We use the Binet formula and the Newton binomial formula to prove the claimed results. Further we present some interesting examples of our results.

1. Introduction

Define second order linear recurrences \( \{U_n\} \) and \( \{V_n\} \) as for \( n > 0 \)
\[ U_n = pU_{n-1} + U_{n-2} \quad \text{and} \quad V_n = pV_{n-1} + V_{n-2}, \]
where \( U_0 = 0 \), \( U_1 = 1 \), and, \( V_0 = 2 \), \( V_1 = p \), respectively. If \( p = 1 \), then \( U_n = F_n \) (\( n \)th Fibonacci number) and \( V_n = L_n \) (\( n \)th Lucas number).

The Binet formulæ are
\[ U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n, \]
where \( \alpha, \beta = \left( p \pm \sqrt{\Delta} \right) / 2 \) and \( \Delta = p^2 + 4 \).

Let \( A(x) \) and \( B(x) \) be the exponential generating functions of sequences \( \{a_n\} \) and \( \{b_n\} \), that is,
\[ A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!} \quad \text{and} \quad B(x) = \sum_{n \geq 0} b_n \frac{x^n}{n!}. \]

The convolution of the exponential generating functions is given as
\[ A(x) B(x) = \sum_{n \geq 0} \left( \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{x^n}{n!}. \]

Many authors have computed various weighted binomial sums by various methods (for more details, see [1, 3, 6, 7, 8, 9, 10, 11]). One of them is to use the convolution of the exponential generating functions and its direct applications, we recall the followings formulæ from the literature (see [3, 10]):
\[ \sum_{i=0}^{n} \binom{n}{i} F_{mi}L_{mn-mi}, \quad \sum_{i=0}^{n} \binom{n}{i} F_{mi}F_{mn-mi}, \]

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Meanwhile, for some integers \( k, t \) and \( r \) such that \( k \neq t \) or \( k \neq r \), the binomial sums

\[
\sum_{i=0}^{n} \binom{n}{i} F_{ti}L_{kni-ri} \quad \text{and} \quad \sum_{i=0}^{n} \binom{n}{i} (-1)^i F_{ti}L_{kni-ri}
\]

can’t be computed by the convolution of exponential generating functions.

Recently, the authors of [4] give general formulæ for the sums

\[
\sum_{i=0}^{n} \binom{n}{i} (n^2 + 1)^{t_i} \sum_{h=0}^{n} \binom{n}{h} h^{m} V_{ht}^{2m+\varepsilon},
\]

where \( t \) is an positive integer and \( \varepsilon \in \{0, 1\} \).

Further the authors of [5] computed the weighted binomial sums

\[
\sum_{k=0}^{n} \binom{n}{k} r_{mk} s_{m(t+n+k)},
\]

where \( r_n \) and \( s_n \) are the terms of \( \{U_n\} \) and \( \{V_n\} \) for some positive integers \( t \) and \( m \). For example, for odd \( m \),

\[
\sum_{i=0}^{n} \binom{n}{i} U_{mi} V_{k+mn+mi} \Delta \binom{n}{i} U_{m} \begin{cases} \binom{n}{i} U_{(k+1)mn} V_{(k+1)mn} & \text{if } n \text{ is even,} \\ \binom{n}{i} U_{(k+1)mn} V_{(k+1)mn} & \text{if } n \text{ is odd,} \end{cases}
\]

and for even \( m \),

\[
\sum_{i=0}^{n} \binom{n}{i} V_{mi} V_{k+mn+mi} = V_{m} U_{(k+1)mn} + 2^n V_{k+mn}.
\]

In this paper we consider new three classes of generalized alternating binomial sums that couldn’t be derived via the convolution of exponential generating functions :

\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i f(n, i, t, k),
\]

where \( f(n, i, k, t) \) will be chosen as \( U_{k+ti} V_{mn-(t+2)ki} \times U_{ki} V_{k+mn} \text{ and } U_{k+ti} V_{mn-(t+2)ki} \times U_{ki} V_{k+mn} \).

These binomial sums (except some special cases of \( k \) and \( t \) ) have not been considered according to our best literature acknowledgement. To compute the claimed sums, our approach is to use Binet formula and the Newton binomial formula. In general, let \( \{A_n\} \) and \( \{B_n\} \) be two second order linear recurrences whose Binet formulæ are

\[
A_n = c_1 a_1^n + c_2 a_2^n \quad \text{and} \quad B_n = d_1 b_1^n + d_2 b_2^n.
\]

Then we write for the sum

\[
\sum_{i=0}^{n} \binom{n}{i} A_i B_{m-bi} = \sum_{i=0}^{n} \binom{n}{i} (c_1 a_1^i + c_2 a_2^i) (d_1 b_1^{m-bi} + d_2 b_2^{m-bi})
\]

\[= \sum_{1 \leq i, j \leq 2} c_i d_j b_j^m (a_i b_j^i + 1)^n.\]

In computing our sums, we will choose required values of the scalers \( a_i, b_i, c_i, d_i \) for \( 1 \leq i \leq 2 \).
2. The main results

First we give a auxiliary lemma and then give our first result.

**Lemma 1.** Let $t$ be any integer. i) For odd $k$,
\[
(-1)^t \alpha^{-k(2t+1)} - \alpha^k = (-1)^{t+1} V_{k(t+1)} \beta^{kt},
\]
\[
(-1)^t \beta^{-k(2t+1)} - \beta^k = (-1)^{t+1} V_{k(t+1)} \alpha^{kt}.
\]

ii) For even $k$,
\[
(\alpha^{-k(2t+1)} - \alpha^k) = -\sqrt{\Delta} U_{k(t+1)} \beta^{kt} \quad \text{and} \quad (\beta^{-k(2t+1)} - \beta^k) = \sqrt{\Delta} U_{k(t+1)} \alpha^{kt}.
\]

**Theorem 1.** For $n > 0$, any integer $t$ and odd $k$,
\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{k(t+2)} V_{kn-ki(t+2)} = (-1)^n V_{k(t+1)}^{n} U_{ktn}
\]
and for even $k$,
\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{k(t+2)} V_{kn-ki(t+2)} = \left\{ \begin{array}{ll}
\Delta^{n-1} \left[ 2U_{k(t+1)}^{n} - U_{k(t+1)}^{n} V_{ktn} \right] & \text{if } n \text{ is odd}, \\
\Delta^{n} U_{k(t+1)}^{n} V_{ktn} & \text{if } n \text{ is even}.
\end{array} \right.
\]

**Proof.** First assume that $k$ is an odd integer. We write by recalling $\alpha \beta = -1$
\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{k(t+2)} V_{kn-ki(t+2)}
\]
\[
= \frac{1}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \left[ (\alpha^{kn-2ki} - \beta^{kn-2ki}) - (-1)^{i} (\alpha^{kn-2ki(t+1)} - \beta^{kn-2ki(t+1)}) \right]
\]
\[
= \frac{\alpha^{kn}}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \alpha^{-2ki} - \frac{\beta^{kn}}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \beta^{-2ki}
\]
\[
- \frac{1}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i(t+1)} (\alpha^{kn-2ki(t+1)} - \beta^{kn-2ki(t+1)})
\]
\[
= \frac{1}{\alpha - \beta} \left[ (\alpha^{k} - \alpha^{-k})^{n} - (\beta^{k} - \beta^{-k})^{n} \right]
\]
\[
- \frac{1}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i(t+1)} (\alpha^{kn-2ki(t+1)} - \beta^{kn-2ki(t+1)})
\]
which, since $\alpha^{k} - \alpha^{-k} = \beta^{k} - \beta^{-k}$ for odd $k$, equals
\[
- \frac{1}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i(t+1)} \left[ (\alpha^{kn-2ki(t+1)} - \beta^{kn-2ki(t+1)}) \right]
\]
\[
= - \frac{1}{\alpha - \beta} \left[ \alpha^{kn} \sum_{i=0}^{n} \binom{n}{i} \left( \alpha^{-2ki(t+1)} - \beta^{kn} \sum_{i=0}^{n} \binom{n}{i} \left( -1 \right)^{i(t+1)} \beta^{-2ki(t+1)} \right) \right]
\]
\[
= - \frac{1}{\alpha - \beta} \left[ \left( \alpha^{k} - (-1)^{t} \alpha^{-k(2t+1)} \right)^{n} - \left( \beta^{k} - (-1)^{t} \beta^{-k(2t+1)} \right)^{n} \right],
\]
which, Lemma 1 (i) and the Binet formula, equals
\[-\frac{1}{\alpha - \beta} \left( (-1)^{tn} V_{k(t+1)}^n \beta^{kn} - (-1)^{tn} V_{k(t+1)}^n \alpha^{kn} \right) = (-1)^{tn} V_{k(t+1)}^n U_{kn},\]
as claimed.

Now we consider the case $k$ is even. We write
\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kt} V_{kn-ki(t+2)}
\]
equal
\[
\frac{1}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \left[ (\alpha^{k(n-2i)} - \beta^{k(n-2i)}) - (\alpha^{kn-2ki(t+1)} - \beta^{kn-2ki(t+1)}) \right],
\]
which, after some arrangements similar to the previous one, equals
\[
\alpha^{kn} (1 - \alpha^{-2k})^n - \beta^{kn} (1 - \beta^{-2k})^n - \alpha^{kn} (1 - \alpha^{-2k(t+1)})^n + \beta^{kn} (1 - \beta^{-2k(t+1)})^n,
\]
which, Lemma 1 (ii), equals
\[
\alpha^{kn} (1 - \alpha^{-2k})^n - \beta^{kn} (1 - \beta^{-2k})^n - \alpha^{kn} (1 - \alpha^{-2k(t+1)})^n + \beta^{kn} (1 - \beta^{-2k(t+1)})^n
\]
equal
\[
\frac{1}{\alpha - \beta} U_k^n \Delta_{\frac{n}{2}} \left[ 1 - (-1)^n \right] + \frac{1}{\alpha - \beta} U_{kt} U_k^n \Delta_{\frac{n}{2}} \left[ (-1)^n \alpha^{kn} - \beta^{kn} \right]
\]
equal
\[
U_k^n \Delta_{\frac{n}{2}} \left[ 1 - (-1)^n \right] + \Delta_{\frac{n-1}{2}} U_{kt} \left[ (-1)^n \alpha^{kn} - \beta^{kn} \right],
\]
which gives the claimed result according to the case of $n$.

For example, we have
\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{10i} V_{5(n-4i)} = V_{15}^n U_{10n}, \quad \sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{-3i} V_{3n-3i} = -V_0^n U_{3n}
\]
and
\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i F_{2i} L_{2(n-3i)} = \begin{cases} 5^{(n-1)/2} (2 - 3^n L_{2n}) & \text{if } n \text{ is odd}, \\ 5^n 3^n F_{2n} & \text{if } n \text{ is even}. \end{cases}
\]

**Lemma 2.** Let $t$ be an integer.

i) For odd $k$,
\[
(-1)^t \alpha^{k(1-2t)} - \alpha^k = (-1)^t U_{kt} \beta^{k(t-1)} \sqrt{\Delta},
\]
\[
(-1)^t \beta^{k(1-2t)} - \beta^k = (-1)^{t+1} U_{kt} \alpha^{k(t-1)} \sqrt{\Delta}.
\]

ii) For even $k$,
\[
\alpha^{k(1-2t)} - \alpha^k = -U_{kt} \beta^{k(t-1)} \sqrt{\Delta}, \quad \beta^{k(1-2t)} - \beta^k = U_{kt} \alpha^{k(t-1)} \sqrt{\Delta}.
\]

**Theorem 2.** For $n \geq 0$ and for odd $k$,
\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kt} V_{k(n-ti)} = U_k^n \begin{cases} (-1)^t V_{kn(t-1)} \Delta_{\frac{n+1}{2}} & \text{if } n \text{ is odd}, \\ U_{kn(t-1)} \Delta_{\frac{n}{2}} & \text{if } n \text{ is even}, \end{cases}
\]
and for even $k$,
\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kt} V_{kn-kti} = U_k^n \begin{cases} -\Delta_{\frac{n-1}{2}} V_{k(t-1)n} & \text{if } n \text{ is odd}, \\ \Delta_{\frac{n}{2}} U_{k(t-1)n} & \text{if } n \text{ is even}. \end{cases}
\]
Proof. First assume that $k$ is odd. We write
\[\sum_{i=0}^{n} \binom{n}{i} (-1)^i U_{kti} V_{kn-kti}\]
\[= \frac{1}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \left[ (\alpha^{kn} - \beta^{kn}) - (-1)^i (\alpha^{kn-2ikt} - \beta^{kn-2ikt}) \right]\]
\[= -\frac{1}{\alpha - \beta} \sum_{i=0}^{n} \binom{n}{i} (-1)^{i(t+1)} (\alpha^{kn-2ikt} - \beta^{kn-2ikt})\]
\[= -\frac{\alpha^{kn}}{\alpha - \beta} (1 + (-1)^{t+1} \alpha^{-2kt})^n + \frac{\beta^{kn}}{\alpha - \beta} (1 + (-1)^{t+1} \beta^{-2kt})^n\]
\[= \frac{1}{\alpha - \beta} \left( \beta^k + (-1)^{t+1} \beta^{k(1-2t)} \right)^n - \frac{1}{\alpha - \beta} \left( \alpha^k + (-1)^{t+1} \alpha^{k(1-2t)} \right)^n,\]
which, by Lemma 2 (ii), equals
\[\frac{1}{\alpha - \beta} U_{kn}^n \Delta^{\frac{n}{2}} \left[ (-1)^t n \alpha^{kn(t-1)} - (-1)^n (-1)^t \beta^{kn(t-1)} \right],\]
which gives the claim according to the case of $n$.

The case $k$ is even could be obtained similarly. \hfill \Box

For example, for $k = 1$, $t = 2$ and the Fibonacci-Lucas case:
\[\sum_{i=0}^{n} \binom{n}{i} (-1)^i F_{2i} L_{n-2i} = \begin{cases} \frac{5^{n+1}}{2} V_n & \text{if } n \text{ is odd}, \\ \frac{5^n}{2} U_n & \text{if } n \text{ is even}. \end{cases}\]

**Lemma 3.**

i) For odd $k$ and $t$,
\[\alpha^{t(k-1)} - \alpha^{t(k+1)} = -V_t \alpha^{tk}, \quad \beta^{t(k-1)} - \beta^{t(k+1)} = -V_t \beta^{tk},\]
\[\alpha^{-t(k+1)} + \alpha^{t(k+1)} = \beta^{-t(k+1)} + \beta^{t(k+1)} = V_{(k+1)t}.\]

ii) For odd $k$ and even $t$,
\[\alpha^{t(k+1)} - \alpha^{t(k+1)} = -U_t \alpha^{tk} \sqrt{\Delta}, \quad \beta^{t(k+1)} - \beta^{t(k+1)} = U_t \beta^{tk} \sqrt{\Delta},\]
\[\alpha^{-t(k+1)} - \alpha^{t(k+1)} = -U_{(k+1)t} \sqrt{\Delta}, \quad \beta^{-t(k+1)} - \beta^{t(k+1)} = U_{(k+1)t} \sqrt{\Delta}.\]

iii) For even $k$ and $t$,
\[\alpha^{t(k-1)} - \alpha^{(k+1)t} = -U_t \alpha^{kt} \sqrt{\Delta}, \quad \beta^{t(k-1)} - \beta^{(k+1)t} = U_t \beta^{kt} \sqrt{\Delta},\]
\[\alpha^{-t(k+1)} - \alpha^{t(k+1)} = -U_{(k+1)t} \sqrt{\Delta}, \quad \beta^{-t(k+1)} - \beta^{t(k+1)} = U_{(k+1)t} \sqrt{\Delta}.\]

iv) For even $k$ and odd $t$,
\[\alpha^{t(k-1)} - \alpha^{(k+1)t} = -V_t \alpha^{kt}, \quad \beta^{t(k-1)} - \beta^{(k+1)t} = -V_t \beta^{kt},\]
\[\alpha^{-t(k+1)} - \alpha^{t(k+1)} = -V_{(k+1)t}, \quad \beta^{-t(k+1)} - \beta^{t(k+1)} = -V_{(k+1)t}.\]

Similar to the previous results, we give the following result without proof.
By Lemma 1 (ii), for even $k$, we have
\[
\alpha^{-k(2t+1)} - \alpha^k = -\sqrt{\Delta U_k(t+1)} \beta^{kt} \quad \text{and} \quad \beta^{-k(2t+1)} - \beta^k = \sqrt{\Delta U_k(t+1)} \alpha^{kt}
\]

and write
\[
\begin{align*}
(1 - \alpha^{-2k(t+1)})^n &= \Delta^{2k} U^n_{k(t+1)} \alpha^{-knt}, \\
(1 - \beta^{-2k(t+1)})^n &= (-1)^n \Delta^{2k} U^n_{k(t+1)} \beta^{-knt},
\end{align*}
\]

or
\[
\begin{align*}
\sum_{i=0}^{n} \binom{n}{i} (-1)^i \alpha^{-2k(t+1)i} &= \Delta^{2k} U^n_{k(t+1)} \alpha^{-knt}, \\
\sum_{i=0}^{n} \binom{n}{i} (-1)^i \beta^{-2k(t+1)i} &= (-1)^n \Delta^{2k} U^n_{k(t+1)} \beta^{-knt}.
\end{align*}
\]

By adding these equalities side by side, we obtain
\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i (\alpha^{-2k(t+1)i} + \beta^{-2k(t+1)i}) = \Delta^{2k} U^n_{k(t+1)} (\alpha^{-knt} + (-1)^n \beta^{-knt} \alpha^{knt}),
\]

or since $V_{-k} = (-1)^k V_k$,
\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^i V_{-2k(t+1)i} = \sum_{i=0}^{n} \binom{n}{i} (-1)^i V_{2k(t+1)i}
\]
\[\begin{align*}
\Delta_n^2 U_{k(t+1)} &= \sum_{i=0}^{n} \binom{n}{i} \left((-1)^i \alpha^{kn} \beta^{-kn} + \alpha^{-kn} \beta^{kn}\right) \\
\Delta_n^2 V_{kn(t+1)} &= \sum_{i=0}^{n} \binom{n}{i} \left((-1)^{k+1} \Delta^{k+1} U_{kn(t+1)}\right)
\end{align*}\]

which, by taking \(m\) instead of \(k(t+1)\) for even \(k\), completes the proof.

Similar to the proof method of Theorem just above, we have the following results without proof by using Lemma 1 (i).

**Theorem 5.** For any integer \(m\) and \(n \geq 0\),

\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^{im} V_{2mi} = (-1)^{mn} V_m V_{nm},
\]

and

\[
\sum_{i=0}^{n} \binom{n}{i} (-1)^{im} U_{2mi} = (-1)^{nm} V_m U_{nm}.
\]

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### References


