A note on the conjecture of Ramirez and Sirvent

Emrah Kılıç
Mathematics Department
TOBB University of Economics and Technology
06560 Ankara
Turkey
ekilic@etu.edu.tr

Helmut Prodinger
Department of Mathematical Sciences
Stellenbosch University
7602 Stellenbosch
South Africa
hproding@sun.ac.za

Abstract
We give a proof of the conjecture of Ramirez and Sirvent [1] on the generating function of the incomplete Tribonacci numbers.

1 Introduction

In a recent paper of Ramirez and Sirvent [1], the authors have defined the incomplete Tribonacci sequence of numbers and polynomials. They have also studied recurrence relations, some properties of these numbers and polynomials, and the generating function of the incomplete Tribonacci numbers.

Let
\[ a_n = \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{n-i-j-1}{i}. \]

Ramirez and Sirvent conjectured that the generating function of the sequence \( \{a_n\}_{n=0}^{\infty} \) is
\[ \sum_{n \geq 0} a_n z^n = \frac{z^3 + z^4}{(1 - z - z^2 - z^3)^2}. \]
In this short note, we give a proof of this conjecture.

2 A proof of the conjecture

Let \( A(z) \) be the generating function of the sequence \( \{a_n\}_{n=0}^{\infty} \). Now consider

\[
A(z) = \sum_{0 \leq j \leq i} i \binom{i}{j} z^{i+j+1} \sum_{n \geq i} \binom{n-i-j-1}{i} z^{n-i-j-1}
\]

\[
= \sum_{0 \leq j \leq i} i \binom{i}{j} z^{i+j+1} \sum_{n \geq i} \binom{n}{i} z^n
\]

\[
= \sum_{0 \leq j \leq i} i \binom{i}{j} z^{i+j+1} \frac{z^i}{(1-z)^{i+1}} = \sum_{0 \leq j \leq i} i \binom{i}{j} \frac{z^{2i+j+1}}{(1-z)^{i+1}}
\]

\[
= \sum_{i \geq 0} i \frac{z^{2i+1}}{(1-z)^{i+1}} \sum_{0 \leq j \leq i} \binom{i}{j} z^j
\]

\[
= \sum_{i \geq 0} i \frac{z^{2i+1}}{(1-z)^{i+1}} (1+z)^i
\]

\[
= \frac{z}{1-z} \sum_{i \geq 0} i \left( \frac{z^2(1+z)}{1-z} \right)^i
\]

\[
= \frac{z}{1-z} \frac{z^2(1-z^2)}{1-z-z^2-z^3}
\]

\[
= \frac{z^3+z^4}{(1-z-z^2-z^3)^2},
\]

which proves the conjecture.

References


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